Rigidity of the chain rule and nearly submultiplicative functions

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Abstract

Assume that $T : C^1(\mathbb{R}) \to C(\mathbb{R})$ nearly satisfies the chain rule in the sense that

$$|T(f \circ g)(x) - (Tf)(g(x))(Tg)(x)| \leq S(x, (f \circ g)(x), g(x))$$

holds for all $f, g \in C^1(\mathbb{R})$ and $x \in \mathbb{R}$, where $S : \mathbb{R}^3 \to \mathbb{R}$ is a suitable fixed function. We show under a weak non-degeneracy and a weak continuity assumption on T that S may be chosen to be 0, i.e. that T satisfies the chain rule operator equation proper, the solutions of which are explicitly known: They are of the form

$$Tf(x) = \frac{H \circ f(x)}{H(x)} |f'(x)|^p \operatorname{sgn} f'(x) \quad f \in C^1(\mathbb{R}), x \in \mathbb{R}$$

where $p > 0$ and H is a strictly positive continuous function on $\mathbb{R}$. We also determine the solutions of one-sided chain rule inequalities like

$$T(f \circ g)(x) \leq (Tf)(g(x))(Tg)(x) + S(x, (f \circ g)(x), g(x))$$

under a further localization assumption. To prove the above results, we investigate the solutions of nearly submultiplicative inequalities on $\mathbb{R}$

$$\phi(\alpha\beta) \leq \phi(\alpha)\phi(\beta) + d \quad \alpha, \beta \in \mathbb{R}$$

under weak restrictions on ϕ. They are multiplicative on $\mathbb{R}_+$, $\phi(\alpha) = \alpha^p$ for $\alpha > 0$ and satisfy $\lim_{\alpha \to \infty} \frac{\phi(-\alpha)}{-\alpha^p} = A \geq 1$, where the limit exists and A might be strictly larger than 1.

We also study the chain rule equation on spaces of real or complex polynomials and entire functions. This is joint work with Vitali Milman.