The Size of Convex Hulls in Hilbert Spaces

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Abstract: Let $A \subset H$ be a compact subset of a (real) Hilbert space H. Denote by

$$\mathrm{co}(A) = \left\{ \sum_{j=1}^{n} \lambda_j x_j : \lambda_j \geq 0\,,\ \sum_{j=1}^{n} \lambda_j = 1\,,\ x_j \in A \right\}$$

the convex hull of A. We will discuss the following problem: How much larger is $\mathrm{co}(A)$ than A? We present several bounds for the size of $\mathrm{co}(A)$, in dependence on that of A. Here the sizes of A and of $\mathrm{co}(A)$ are measured by their entropy numbers $e_n(A)$ or, equivalently, by their covering numbers $N(A, \varepsilon)$.

Roughly spoken, the results assert the following: If A is "small", then $\mathrm{co}(A)$ can be much bigger, but its size cannot exceed a certain critical bound. On the other side, if A is already "big", then $\mathrm{co}(A)$ has about the same size as A.

Of special interest is the critical case which corresponds to $e_n(A) \approx n^{-1/2}$ or, equivalently, to $\ln N(A, \varepsilon) \approx \varepsilon^{-2}$. No geometrical explanation for the existence of this critical case is known. But there exists a probabilistic reason in terms of Gaussian random processes. We will discuss this relation in detail.

Finally, if time admits, we present some applications of the results. They allow us to estimate the entropy numbers for certain summation operators on binary trees.