

The Accelerating Universe and Vacuum Energy

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Abstract

Different theories have been proposed to explain the current acceleration of the Universe by screening the effect of the predicted, but not observed, large vacuum energy. One such theory is the Fab Four which uses a scalar function model that dynamically adapts for a large vacuum energy. The theory has been shown to screen the predicted vacuum energy, but little work has been done to check its stability. Through this paper we checked the stability of the late time attractor of flat space using perturbations. We found that using different combinations of Lagrangians found in the Fab Four could result in an instability in the late time attractor shedding doubts on the validity of the Fab Four.

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1 Introduction

For around a decade we have known that the expansion of the universe is accelerating. The problem is that we do not know why it is accelerating at its current rate or what exactly the dark energy is that drives it. The expected vacuum energy should be much larger and causing more acceleration than we see. An idea came about that the acceleration could be caused by a modification to General Relativity instead of being caused by an exotic form of energy. One of the possibilities of these modifications has been adding a scalar field to the tensors already in General Relativity. Normally General Relativity is only a tensor theory, but the addition of a scalar field can lead to some interesting consequences. This scalar field can be made to screen the effects of a large cosmological constant and prevent the universe from expanding faster than we previously thought it would.

In [1] G. Horndeski derived what is thought to be the most general case of a scalar-tensor theory with second order field equations in four dimensions. He started with a Lagrangian that was a function of the metric, a scalar field and derivatives of the metric and scalar field including derivatives greater than second order. With the defined Lagrangian and its dependencies, he was able to derive an action for the most general scalar-tensor theory with second order field equations in four dimensions. Recent work has used G. Horndeski's action to derive a new class of models, called the Fab Four [2, 3], on the Friedmann-Lemaitre-Robertson-Walker (FLRW) metric [4], which describes a homogenous, isotropic expanding or contracting universe.

The Fab Four are a set of four Lagrangians that lead to a self-tuning of the cosmological constant when used together. This self-tuning causes solutions of the equations of motion to approach that of flat space given any initial cosmological constant and the self-tuning remains there even when the cosmological constant goes through a phase transition. The self-tuning must also permit non-trivial cosmologies, so the only possible cosmology is not flat space. Some work has been done on the Fab Four in [5] to derive some cosmological solutions for different conditions. There are solutions for inflationary, radiation and matter

dominated epochs. The problem is that none of these solutions have had their stabilities tested.

If such a theory works for explaining the current state of the universe from earlier conditions, then it is possible that it could solve the problem of where the acceleration of the universe actually comes from. This theory would lead to further understanding of the universe and could help predict whats in store for the universe.

To be a viable theory, the solutions to the equations of motion have to be stable after they reach flat space. If the solutions are unstable once they reach flat space then any perturbation could cause the system to spiral out of control. The goal of the project is to test the stability of the solutions obtained. We will need to check that the solutions will return to that of flat space after being perturbed. This work will involve the derivation of the equations that govern the evolution of the cosmology in different models and then the examination of each solution individually. If the solutions are unstable then it will require that some changes need to be made to the Fab Four to prevent them from being unphysical.

1.1 Horndeski's Lagrangian

Horndeski's most general second order scalar-tensor theory yields a Lagrangian that at most only contains two derivatives of the scalar field. Considering that Lagrangian originally included higher order derivatives this result is nice. The Lagrangian we obtain from Horndeski's theory is much better than only having second order derivatives. In general, a Lagrangian with second order derivatives will lead to equations of motion with fourth order derivatives. Taking the second order Euler-Lagrange equation

$$\frac{d^2}{dt^2} \left(\frac{\partial \mathcal{L}}{\partial \ddot{x}} \right) - \frac{d}{dt} \left(\frac{\partial \mathcal{L}}{\partial \dot{x}} \right) + \frac{\partial \mathcal{L}}{\partial x} = 0 \quad (1)$$

where \dot{x} represents a time derivative of x , and a test Lagrangian

$$\mathcal{L} = \frac{\ddot{x}^2}{a^2} + \dot{x}^2 + m^2 x^2 \quad (2)$$

will yield the the equation of motion

$$\frac{d^4 x}{dt^4} - a^2 \ddot{x} + a^2 m^2 x = 0 \quad (3)$$

Fourth order differential equations require four initial conditions. The evolution of every single particle (or physical degree of freedom) is determined (classically) by two initial conditions. The presence of four initial conditions implies that there are really two physical degrees of freedom propagating in this system, this means there is one extra particle compared to what we would have had if the Lagrangian only involved second order equations of motion. By taking (3) and going into Fourier space we can obtain a propagator

$$G = \frac{1}{w^2 - a^2 w^2 + a^2 m^2} \quad (4)$$

where w is the momentum in Fourier space, a is a scalar constant and m is the mass of the particle this is representing. Further work can be done on (4) to show that it is the same as

$$G = \frac{A}{w^2 - m_1^2} - \frac{B}{w^2 - m_2^2} \quad (5)$$

where w is the same as before, but A , B , m_1 and m_2 are positive valued functions of a and m from (4). Instead of being the propogator for one particle, (5) is the propagator for two sperate particles. Because A and B are positive valued and there is a minus sign in front of B it means that the energy of this two particle system is not bounded from below. The two particle state is unstable and has infact an Ostrogadsky instability and a ghost. See Appendix A for a more detailed dervation of the propator and its decomposition.

Although the Lagrangian that Horndeski found contains second derivatives it happens

to be ghost free. Using the Lagrangian will not lead to the previously described instability. Because the Lagrangian is free from this Ostrogradski instability it allows it to be used and modified without having to worry about its initial stability. Modifications can, and as we will show in the case of the Fab Four, effect the stability of Horndeski's Lagrangian, but it is allowed to be a stable starting position for any theory to build off of.

1.2 Fab Four

The Fab Four is first formed by taking Horndeski's Lagrangian on Friedmann-Lemaitre-Robertson-Walker (FLRW) backgrounds with a homogenous scalar $\phi = \phi(t)$ that depends only on time. An FLRW background is a homogenous isotropic expanding or contracting solution with to Einstein's field equations of General Relativity that also contains spatial curvature. Applying a set of three self-tuning criteria to this Lagrangian is what actually gives the Fab Four. These three criteria are

1. Solutions must approach that of flat space for any value of net cosmological constant
2. The previous criteria must remain true before and after any phase transition where the cosmological constant jumps instantaneously by a finite amount
3. The theory should permit a non-trivial cosmology

The first criteria means that any given starting configuration will eventually turn into flat space even if there is a cosmological constant involved. The second one allows for the possibility of phase transitions. This would allow for different epochs of the universe such as inflation or the matter dominated era. The third one is to make sure that the theory will yield more than starting in flat space and staying that way. More complicated solutions have to exist. Applying the self-tuning criteria to Horndeski's Lagrangian leaves only four terms that depend on an arbitrary scalar and other terms that can be determined from a metric. Each term contains an arbitrary function of the scalar and possibly derivatives of the scalar. The function of the scalar becomes the potential associated with each term of

the Fab Four. A side effect of the self-tuning is that the theory does not work for positive spatial curvature. Combining all four term yields the Fab Four. All four terms are taken together because self-tuning does not work on them individually. One can recover General Relativity from the Fab Four by setting the scalar field to a constant.

The general form of the Fab Four is given by

$$\mathcal{L}_{george} = \mathcal{L}_1 = \sqrt{-g}V_{john}(\phi)R \tag{6}$$

$$\mathcal{L}_{ringo} = \mathcal{L}_2 = \sqrt{-g}V_{ringo}(\phi)\hat{G} \tag{7}$$

$$\mathcal{L}_{paul} = \mathcal{L}_3 = \sqrt{-g}V_{paul}(\phi)P^{\mu\nu\alpha\beta}D_\mu\phi D_\alpha\phi D_\nu D_\beta\phi \tag{8}$$

$$\mathcal{L}_{john} = \mathcal{L}_4 = \sqrt{-g}V_{john}(\phi)G^{\mu\nu}D_\mu\phi D_\nu\phi \tag{9}$$

where R is the scalar curvature, $\sqrt{-g}$ is the square root of the determinant of the metric, \hat{G} is the Gauss-Bonnet combination, $P_{\mu\nu\alpha\beta}$ is the Riemann double dual tensor, $G^{\mu\nu}$ is the Einstein tensor, ϕ is the scalar in this scalar-tensor theory, D_β is a covariant derivative, and $V_*(\phi)$ is the potential as a function of ϕ associated with each of the Fab Four.

There has been a group that has taken the Fab Four further and generated a Fab 5 in [6]. This Fab 5 adds a nonlinear term that combines the kinetic gravity terms to the Fab Four. Adding this new term doesnt affect the self-tuning features of the Fab Four. More work was done with the Fab 5 in [7] that showed that they cant be self-tuning, stable, and have flat space as an attractor solution in the future and past. The Fab 5 can only have two of the three previously mentioned properties at a time. To be a viable, we would expect all three at once, so it prevents the Fab 5 from being a practical theory. While the Fab 5 have been shown to not be a viable theory, there has been little previous work testing the stability of

the Fab Four to see if it is stable.

2 Objectives

The purpose of this work was to test the stability of the Fab Four. First we wanted an explicit form of the Fab Four on the FLRW metric. With the explicit form, we wanted to obtain background solutions. We were really looking for a Friedmann like equation, an equation of motion for the scalar field and the Hamilton. Our next goal objective was to rederive the previous equations after perturbing the scalar field and the metric. From the perturbed Fab Four we wanted to check the stability of the theory.

3 Methods

3.1 Background Solutions

The first thing to be done was to find the explicit form of the Fab Four on the FLRW metric given by

$$g_{\mu,\nu} = \begin{pmatrix} -N^2 & 0 & 0 & 0 \\ 0 & \frac{a^2}{1-kr^2} & 0 & 0 \\ 0 & 0 & r^2 a^2 & 0 \\ 0 & 0 & 0 & a^2 r^2 \sin^2(\theta) \end{pmatrix} \quad (10)$$

with distances given by

$$ds^2 = N(t)^2 dt^2 - a^2(t) \gamma_{ij} dx^i dx^j \quad (11)$$

where N is the lapse, which relates the flow of proper time to actual time, a is the scale factor, which describes the relative expansion of the universe and k is the spatial curvature of the universe. Usually the lapse is set to one, but it is usefull to keep it in for now and set its

value after we find some background equations. The FLRW metric describes a homogenous, isotropic expanding universe with spatial curvature.

This was done through standard tensor calculus to derive values that were based on the metric and derivatives of the metric such as the scalar curvature, Gauss-Bonnet combination, Riemman double dual and Einstien Tensor. The result was.

$$\mathcal{L}_{george} = \mathcal{L}_1 = \frac{6 \sin(\theta)r^2}{\sqrt{1-kr^2}} \left((aNk - \frac{a\dot{a}^2}{N})V_1(\phi) - \frac{a^2\dot{a}}{N}V_1'(\phi)\dot{\phi} \right) \quad (12)$$

$$\mathcal{L}_{ringo} = \mathcal{L}_2 = \frac{-24 \sin(\theta)r^2}{\sqrt{1-kr^2}} \left(\frac{\dot{a}^3}{3N^3} + \frac{k\dot{a}}{N} \right) V_2'(\phi)\dot{\phi} \quad (13)$$

$$\mathcal{L}_{paul} = \mathcal{L}_3 = \frac{\sin(\theta)r^2}{\sqrt{1-kr^2}} \frac{3\dot{a}}{N^3} \left(\frac{\dot{a}^2}{N^2} + k \right) V_3(\phi)\dot{\phi}^3 \quad (14)$$

$$\mathcal{L}_{john} = \mathcal{L}_4 = \frac{\sin(\theta)r^2}{\sqrt{1-kr^2}} \frac{3a}{N} \left(\frac{\dot{a}^2}{N^2} + k \right) V_4(\phi)\dot{\phi}^2 \quad (15)$$

We used $V_*'(\phi)$ to represent $\frac{dV_*(\phi)}{d\phi}$. For the rest of the paper we adopt the notation of referring to the different Lagrangians by their numbers in (12)-(15). Now we can combine all of the Fab Four and a term containing the energy density $\rho(a)$ that would drive expansion to form

$$\mathcal{L} = \mathcal{L}_1 + \mathcal{L}_2 + \mathcal{L}_3 + \mathcal{L}_4 - \sqrt{-g}\rho(a) \quad (16)$$

$$\begin{aligned} \mathcal{L} = & \frac{6 \sin(\theta)r^2}{\sqrt{1-kr^2}} \left[(aNk - \frac{a\dot{a}^2}{N})V_1(\phi) - \frac{a^2\dot{a}}{N}V_1'(\phi)\dot{\phi} - \right. \\ & 4 \left(\frac{\dot{a}^3}{3N^3} + \frac{k\dot{a}}{N} \right) V_2'(\phi)\dot{\phi} + \frac{\dot{a}}{2N^3} \left(\frac{\dot{a}^2}{N^2} + k \right) V_3(\phi)\dot{\phi}^3 + \\ & \left. \frac{a}{2N} \left(\frac{\dot{a}^2}{N^2} + k \right) V_4(\phi)\dot{\phi}^2 - \rho(a)a^3N \right] \quad (17) \end{aligned}$$

This new \mathcal{L} represents the Lagrangian for the entire system. With the normal Lagrangian formed from the FLRW metric we could find the Friedmann Equation by using the Euler-Lagrange Equation with respect to the lapse. We do the same with our Lagrangian \mathcal{L} and find it contains no \dot{N} terms so we wind up with only

$$\frac{\partial \mathcal{L}}{\partial N} = 0 \quad (18)$$

or

$$\begin{aligned} H^2[V_1(\phi) + \frac{1}{NH}V_1'(\phi)\dot{\phi} + \frac{4}{N}HV_2'(\phi)\dot{\phi} - \frac{5H}{2N^2}V_3\dot{\phi}^3 - \frac{3}{2N^2}V_4(\phi)\dot{\phi}^2] = \\ \rho(a) - \frac{k}{a^2}[V_1(\phi) + \frac{4}{N}HV_2'(\phi)\dot{\phi} - \frac{3H}{2N^3}V_3'(\phi)\dot{\phi}^3 - \frac{1}{2N^3}V_4(\phi)\dot{\phi}^2] \end{aligned} \quad (19)$$

where H is $\frac{\dot{a}}{a}$, otherwise known as the Hubble parameter. We can compare our Friedmann Equation to the original one to see some effects of the Fab Four. Our units are set such that the speed of light $c = 1$ and $\frac{8\pi G}{3} = 1$.

$$H^2 = \rho(a) - \frac{k}{a^2} \quad (20)$$

In the left hand side of (20), H^2 describes the acceleration of the expansion of the universe while the right hand side of the equation describes what is causing that acceleration. What is predicted is a large $\rho(a)$, which is the energy density, but a much smaller one is being observed. The $\frac{k}{a^2}$ term is expected to cancel out some of the energy density, but the fact that positive curvature is needed for self tuning means that there is actually nothing there to oppose the energy density. If we now look at (19) we see that there is now a factor dependent on ϕ that can be used to increase the effect of $\frac{k}{a^2}$. The potentials V_1 through V_4 can be formulated such that their effect allows the right hand side of (19) to yield the observed value of a slower than predicted expansion.

Because we know what the end state of any system under the Fab Four is a flat one, we can write down the Friedmann Equation for the late time attractor, which is only flat space under a possible net cosmological constant. To describe this we set the scale factor a to a constant. Having a as a constant means that $\dot{a} = 0$ and $H = 0$. Our Friedmann Equation for the late time attractor is then

$$\frac{a^2}{k}\rho(a) = V_1(\phi) - \frac{1}{2}V_4(\phi)\dot{\phi}^2 \quad (21)$$

Another usefull equation to have was the background equation of motion for the scalar field ϕ . We obtained this by applying the Euler-Lagrange equations again, but this time with respect to ϕ . After setting the lapse equal to one we obtain

$$\begin{aligned} & ([2\dot{a}k + 3\dot{a}^3]V_3(\phi)\dot{\phi} + [a\dot{a}^2 + ak]V_4(\phi) + [-a\dot{a}^2 - a^2\ddot{a} - ak]V_1'(\phi))\ddot{\phi} + \\ & \quad [-4\dot{a}^2\ddot{a} - 4k\ddot{a}]V_2'(\phi) + [-3k\dot{a}]V_2''(\phi)\dot{\phi} + [\frac{9}{2}\dot{a}^2\ddot{a} + \frac{3}{2}\ddot{a}k]V_3(\phi)\dot{\phi}^2 + \\ & \quad [\dot{a}^3 + \dot{a}k]V_3'(\phi)\dot{\phi}^3 + [\dot{a}^3 + 2a\dot{a}\ddot{a} + \dot{a}k]V_4(\phi)\dot{\phi} - \frac{1}{2}[\dot{a}^2 + ak]V_4'(\phi)\dot{\phi}^2 = 0 \end{aligned} \quad (22)$$

In its current form, (22) is not very useful. The four different potentials V_* are yet left arbitrary and leave no analytic solution to the equation of motion for ϕ . We re-write this in during our late term attractor phase to get

$$\ddot{\phi} - \frac{V_4'(\phi)}{2(V_4(\phi) - V_1'(\phi))}\dot{\phi}^2 = 0 \quad (23)$$

We see that (23) is much easier to solve than (22) and involves only two of the potentials. $V_1(\phi)$ and $V_4(\phi)$ are still arbitrary functions of ϕ and can take different forms. If we were to look for solutions that satisfy the Fab Four than (23) would be a good place to start, but our main goal is to look at the stability of the theory as a whole and not individual solutions.

If we consider a and ϕ as the dynamic quantities in the Fab Four then we can define the

Hamiltonian as follows.

$$\mathcal{H} = p_a \dot{a} + p_\phi \dot{\phi} - \mathcal{L} \quad (24)$$

which gives an explicit form of

$$\begin{aligned} \mathcal{H} = \frac{6 \sin(\theta) r^2}{\sqrt{1 - kr^2}} & [-(a\dot{a}^2 - ak)V_1(\phi) - a^2 \dot{a} V_1'(\phi) \dot{\phi} - 4(\dot{a}^2 + k\dot{a})V_2' \dot{\phi} + \\ & (\frac{5}{2}\dot{a}^3 + \frac{3}{2}k\dot{a})V_3 \dot{\phi}^3 + (\frac{3}{2}a\dot{a}^2 + \frac{1}{2}ak)V_4 \dot{\phi}^2 + \rho(a)a^3] \end{aligned} \quad (25)$$

As with the equation of motion, the Hamiltonian would be useful for finding and analyzing specific solutions of the Fab Four.

3.2 Perturbations

Now that the background equations have been obtained we can work on actually testing the stability of the system. To test this we add scalar perturbations into both the scalar field and the metric. Before the scalar field depended only on time, but now we add a perturbation that depends on both time and space. The metric also gets two perturbations that depend on both time and space. One affects the time component of the metric, while the other one affects the spatial components. Our perturbations take the form of

$$\phi \rightarrow \bar{\phi}(t) + \epsilon \delta\phi(t, r, \theta, \varphi) \quad (26)$$

$$\begin{aligned} ds^2 &= N(t)^2 dt^2 - a^2(t) \gamma_{i,j} dx^i dx^j \rightarrow \\ ds^2 &= N(t)^2 (1 + 2\epsilon \xi(t, r, \theta, \varphi)) dt^2 - (1 - 2\epsilon \psi(t, r, \theta, \varphi)) a^2(t) \gamma_{i,j} dx^i dx^j \end{aligned} \quad (27)$$

These perturbations represent the degrees of freedom in our system. In normal General

Relativity, where we don't have our scalar field, the perturbations on the metric ξ and ψ would be the normal two degrees of freedom of a system. We want to check and make sure that the perturbations evolve in time and don't grow infinitely.

Now that the perturbations have been added, most work must now be done in Mathematica. Even keeping to only second order in perturbations in the Lagrangian is too unyielding to do by hand, so we were assisted by Ricci.m in our calculations. It is even too large to bother putting into this paper.

With the Fab Four recreated with perturbations we want a way of looking at the perturbations and seeing their behavior. One way is to look at the an-isotropic stress in the Stress-Energy Tensor. We can find the Stress-Energy Tensor by varying the Action. The action for the Fab Four is given by

$$S = \int -\sqrt{-g}M_{pl} \mathcal{L} dx^4 \quad (28)$$

where M_{pl} is the Planck Length. Now that we have the Action, we can vary it with respect to the metric to get

$$\frac{1}{\sqrt{-g}} \frac{\delta S}{\delta g^{\mu\nu}} = 0 \quad (29)$$

and

$$\frac{1}{\sqrt{-g}} \frac{\delta S}{\delta g^{\mu\nu}} = M_{pl} \frac{1}{\sqrt{-g}} \frac{\delta}{\delta g^{\mu\nu}} (-\sqrt{-g} \mathcal{L}) = T_{\mu\nu} \quad (30)$$

with $T_{\mu\nu}$ being the Stress-Energy Tensor. Because varying with respect to the metric is linear, we can split up the variation of the Lagrangian to be done on each of the individual Fab Four. Each one will give their own Stress-Energy tensor that can be added together to form the total one.

$$\begin{aligned}
& \frac{1}{\sqrt{-g}} \frac{\delta}{\delta g^{\mu\nu}} (\sqrt{-g} \mathcal{L}_1) + \frac{1}{\sqrt{-g}} \frac{\delta}{\delta g^{\mu\nu}} (\sqrt{-g} \mathcal{L}_2) + \frac{1}{\sqrt{-g}} \frac{\delta}{\delta g^{\mu\nu}} (\sqrt{-g} \mathcal{L}_3) + \frac{1}{\sqrt{-g}} \frac{\delta}{\delta g^{\mu\nu}} (\sqrt{-g} \mathcal{L}_4) \\
& = T_{\mu\nu} = T_{\mu\nu}^{(1)} + T_{\mu\nu}^{(2)} + T_{\mu\nu}^{(3)} + T_{\mu\nu}^{(4)}
\end{aligned} \tag{31}$$

Using modified forms of what is found in [8] and [9], we can find actual forms of the individual Stress-Energy Tensors.

$$T_{\mu\nu}^{(1)} = \square \tilde{V}_1 g_{\mu\nu} - \Pi_{\mu\nu}^{(1)} + \tilde{V}_1 G_{\mu\nu} \tag{32}$$

$$\begin{aligned}
T_{\mu\nu}^{(2)} = & \frac{1}{2} g_{\mu\nu} \tilde{V}_2 \hat{G} - 2g_{\mu\alpha} g_{\nu\beta} \tilde{V}_2 (R R^{\alpha\beta} - 2R_\rho^\alpha R^{\beta\rho} + R^{\alpha\rho\theta\tau} R_{\rho\theta\tau}^\beta + 2R^{\alpha\rho\theta\beta} R_{\rho\theta}) + \\
& 2g_{\mu\alpha} g_{\nu\beta} ((D^\alpha D^\beta \tilde{V}_2) R - g^{\alpha\beta} (\square \tilde{V}_2) R - 2(D_\rho D^\alpha \tilde{V}_2) R^{\beta\rho} - 2(D_\rho D^\beta \tilde{V}_2) R^{\alpha\rho} + \\
& 2(\square \tilde{V}_2) R^{\alpha\beta} + 2g^{\alpha\beta} (D_\rho D_\theta \tilde{V}_2) R^{\rho\theta} - 2(D_\rho D_\theta \tilde{V}_2) R^{\alpha\rho\theta\beta})
\end{aligned} \tag{33}$$

$$\begin{aligned}
T_{\mu\nu}^{(3)} = & 6(\Pi_{\mu\nu}^{(3)})^2 - 6[\Pi^{(3)}](\Pi_{\mu\nu}^{(3)})^2 + 3([\Pi^{(3)}]^2 - [(\Pi^{(3)})^2])\Pi_{\mu\nu}^{(3)} - \\
& g_{\mu\nu}([\Pi^{(3)}]^3 - 3[(\Pi^{(3)})^2][\Pi^{(3)}] + 2[(\Pi^{(3)})^3]) + \frac{3}{2} L_{\mu\alpha\nu\beta} \Pi^{\mu\nu} (\partial \tilde{V}_3)^2
\end{aligned} \tag{34}$$

$$T_{\mu\nu}^{(4)} = (\Pi_{\mu\nu}^{(4)})^2 - (\square \tilde{V}_4) \Pi_{\mu\nu}^{(4)} - \frac{1}{2} ([(\Pi^{(4)})^2] - [\Pi^{(4)}]^2) g_{\mu\nu} + \frac{1}{2} L_{\mu\alpha\nu\beta} \partial^\alpha \tilde{V}_4 \partial^\beta \tilde{V}_4 + \frac{1}{2} G_{\mu\nu} (\partial \tilde{V}_4)^2 \tag{35}$$

Where

$$\begin{aligned}
\Box &= g^{\mu\nu} \partial_\mu \partial_\nu \\
L^{\mu\alpha\nu\beta} &= 2R^{\mu\alpha\nu\beta} + 2(R^{\mu\beta} g^{\nu\alpha} + R^{\nu\alpha} g^{\mu\beta} - R^{\mu\nu} g^{\alpha\beta} - R^{\alpha\beta} g^{\mu\nu}) + R(g^{\mu\nu} g^{\alpha\beta} - g^{\mu\beta} g^{\nu\alpha}) \\
(\partial\tilde{V}_i)^2 &= g^{\mu\nu} \partial_\mu \tilde{V}_i \partial_\nu \tilde{V}_i \\
\hat{G} &= R^2 - 4R_{\mu\nu} R^{\mu\nu} + R_{\mu\nu\alpha\beta} R^{\mu\nu\alpha\beta} \\
\Pi_{\mu\nu}^{(i)} &= D_\mu D_\nu \tilde{V}_i \\
(\Pi_{\mu\nu}^{(i)})^2 &= g^{\alpha\beta} \Pi_{\mu\alpha}^{(i)} \Pi_{\beta\nu}^{(i)} \\
(\Pi_{\mu\nu}^{(i)})^3 &= g^{\alpha A} g^{\beta B} \Pi_{A\beta}^{(i)} \Pi_{\mu\alpha}^{(i)} \Pi_{B\nu}^{(i)} \\
[(\Pi^{(i)})^2] &= \Pi_{\mu\nu}^{(i)} (\Pi^{(i)})^{\mu\nu} \\
[(\Pi^{(i)})^3] &= g^{\mu\nu} (\Pi_{\mu\nu}^{(i)})^3 \\
[\Pi^{(i)}]^2 &= (\Pi^{(i)})_\mu^\mu (\Pi^{(i)})_\nu^\nu \\
[\Pi^{(i)}]^3 &= (g^{\mu\nu} \Pi_{\mu\nu}^{(i)})^3
\end{aligned}$$

and

$$\tilde{V}_1 = V_1(\bar{\phi}) \quad (36)$$

$$\tilde{V}_2 = V_2(\bar{\phi}) \quad (37)$$

$$\tilde{V}_3 = (V_3')^3(\bar{\phi}) \quad (38)$$

$$\tilde{V}_4 = (V_4')^2(\bar{\phi}) \quad (39)$$

We can separate terms in the Stress-Energy tensor based off of their order of perturbations to get

$$T_{\mu\nu} = \bar{T}_{\mu\nu} + \epsilon \delta T_{\mu\nu} + O(\epsilon^2) \quad (40)$$

$\bar{T}_{\mu\nu}$ will satisfy the background equation and should be zero. Now we will only focus on the first order of perturbations and raise the first index with the inverse metric to obtain

$$0 = \delta T_{\nu}^{\mu} = \delta T^{(1)\mu}_{\nu} + \delta T^{(2)\mu}_{\nu} + \delta T^{(3)\mu}_{\nu} + \delta T^{(4)\mu}_{\nu} \quad (41)$$

We are interested in the stability of the late-time attractor, so we can use (21) and (41) to find how the perturbations evolve in time after a flat solution is obtained. The anisotropic stress from the Stress-Energy tensor will give us relations between the different perturbations and (21) allows one to eliminate some of the terms in the Stress-Energy tensor. By using the T_r^t , T_{φ}^t , and T_r^r components of the Stress-Energy tensor and (21) we can get the relations that we want.

4 Results

As a test, we wanted to see what would happen if we only had V_1 and all other potentials were zero. In other words we are testing General Relativity. Under these assumptions (21) becomes

$$\frac{a^2}{k}\rho(a) = \tilde{V}_1 \quad (42)$$

We see that V_1 is a constant and when we work through to find equations governing ψ and ξ we find that $\psi = \xi$ and that they are both a constant. ϕ and ξ are supposed to be degrees of freedom, but because there are no dynamics at all they aren't degrees of freedom. This loss of two degrees of freedom means that the Fab Four aren't stable when only V_1 is non-zero. A result like that is not unexpected considering that using only V_1 yields no self-tuning.

We also checked to see what would happen if only V_4 was non-zero. This case seems as a possibility for something interesting to happen as V_4 is one of the two potential in the late

time attractor (21). For this we can rewrite (21) with

$$\frac{a^2}{k}\rho(a) = -\frac{1}{2}(\tilde{V}'_4)^2\dot{\phi}^2 \quad (43)$$

After analyzing the Stress-Energy Tensor in the case of (43) we find that

$$\delta\phi = \psi = \xi = 0 \quad (44)$$

All of the perturbations have dissappeared. We should expect something bad like this to happen as having only V_4 is not that physical of a system. It certainly doesn't give self-tuning on its own and doesn't represent General Relativity or any other standard model alone.

If we now look at what happens when we have \tilde{V}'_1 and \tilde{V}'_2 and (21) in its entirety, we have our first case of something that could self-tune. The problem is that when we solve for ψ , we find it has the form of

$$\dot{\psi} + (k^2 - m^2)\psi = 0 \quad (45)$$

where k is the momentum in Fourier space and m is a scalar. In normal General Relativity we would find a form of

$$\ddot{\psi} + (k^2 + m)\psi = 0 \quad (46)$$

for ψ . The difference between (45) and (46) is that ψ in (46) has two time derivatives while it has one in (45). We can solve (45) to get

$$\psi = Ae^{-(k^2-m^2)t} \quad (47)$$

It's possible that the perturbations are unbounded in this case. Having no bound would allow the perturbations to grow infinitely. Looking at (3.2), this makes the distance between any two points infinite. We are also dealing with the fact that we lost the kinetic term on

what we thought was a degree of freedom. ψ is not dynamical anymore. The value of ψ can be solved for as it has in (47), so it is not independent as the ψ from (46) would be. With all of this going on the quantum corrections at this point run out of control.

What we see is that the flat solution to the Fab Four is not actually stable and physical. The main problem with this is that the flat solution is the late time attractor and all solutions of the Fab Four will eventually become flat. It is supposed to be the end state. There will be self-tuning that will bring everything to a flat solution, but they won't stay there. They will try to find a more stable solution. Without the late time attractor, the Fab Four can't be stable. It is possible that the addition of \tilde{V}_2 and \tilde{V}_3 could fix the stability issue, but it seems unlikely.

The instability that we are looking at happens when using only \tilde{V}_1 and \tilde{V}_4 and both simultaneously. It is believed that the same thing would happen if one were to include \tilde{V}_2 and \tilde{V}_3 into the calculations. Time constraints prevented further analysis.

For a more detailed derivation of the equations describing the perturbations please see Appendix B

5 Conclusions

While checking the stability of the Fab Four, in the late time attractor phase, we found what was found in [2], that using only one of the four potentials doesn't work. In the case of only V_1 or V_4 we wound up with constant perturbations or no perturbations at all. When we did consider a case with V_1 and V_4 , we found the form of one of the metric perturbations ψ given by (45). The problem was that the perturbation was unbounded and it led to instability. Because the late time attractor appears to be unstable, it sheds doubt onto the validity of the Fab Four as a theory. More work would need to be done to check and see if the additon of \tilde{V}_2 and \tilde{V}_3 would make it stable, but as said before, it seems unlikely that they would fix the instability.

Appendix A

Green's Function, Propogator and Ostrogadsky Instability

To get the propagator out of (3) and give it a source J and use $\phi(t)$ to represent the particle instead of x .

$$\frac{d^4\phi(t)}{dt^4} + a^2\ddot{\phi}(t) + a^2m^2\phi(t) = J \quad (\text{A.1})$$

We know transform the function into Fourier Space using

$$\phi(t) = \int dw \phi(w)e^{iwt} \quad (\text{A.2})$$

$$J = \int dw J_w e^{iwt} \quad (\text{A.3})$$

where w is the momentum in Fourier Space and J_w is the Fourier Transform of J . Now applying (A.2) and (A.3) to (A.1) will yield

$$\int dw i^4 \phi(w)e^{iwt} + \int dw i^2 a^2 \phi(w)e^{iwt} + \int dw a^2 m^2 \phi(w)e^{iwt} = \int dw J_w e^{iwt} \quad (\text{A.4})$$

Taking the derivative with respect to w and then dividing out e^{iwt} leaves

$$(w^4 - a^2w^2 + a^2m^2)\phi(w) = J_w \quad (\text{A.5})$$

or

$$\phi(w) = \frac{J_w}{(w^4 - a^2w^2 + a^2m^2)} \quad (\text{A.6})$$

To make sense of (A.6) we first need to know what a Green's function is. For a linear differential operator L and

$$Lu(x) = f(x) \quad (\text{A.7})$$

a Green's function $G(x, s)$ is defined as

$$u(x) = \int ds G(x, s) f(s) \quad (\text{A.8})$$

Where $f(s)$ is the Fourier transform of $f(x)$. Now if we use

$$L = \left(\frac{d^4}{dt^4} + a^2 \frac{d^2}{dt^2} + a^2m^2 \right) \quad (\text{A.9})$$

and

$$Lu(t) = J \quad (\text{A.10})$$

we can rewrite (A.6) as

$$\phi(t) = \int dw \phi(w) e^{iwt} = \int dw \frac{J_w}{(w^4 - a^2w^2 + a^2m^2)} e^{iwt} \quad (\text{A.11})$$

From this we can see that the Green's function

$$G(t, w) = \frac{e^{iwt}}{(w^4 - a^2w^2 + a^2m^2)} \quad (\text{A.12})$$

In field theory we would call this the propagator. The propagator can be seen as the inverse of the waver operator which is a linear differential operator, so it is infact a Green's function. We can also ignore the e^{iwt} as it is only a phase that won't effect the dynamics of particle. This work has led us to obtaining the propagator

$$G(w) = \frac{1}{(w^4 - a^2w^2 + a^2m^2)} \quad (\text{A.13})$$

We can modify the the propagator in (A.13) to be the propagator for two seperate particles of the form

$$G(w) = \frac{A}{w^2 - m_1^2} + \frac{B}{w^2 - m_2^2} \quad (\text{A.14})$$

Setting (A.13) equal to (A.14) gives a system of equations that can be used to solve for A , B , m_1 and m_2 .

$$(A + B)w^2 = 0 \quad (\text{A.15})$$

$$-(Am_2^2 + Bm_1^2) = 1 \quad (\text{A.16})$$

$$a^2 = (m_1^2 + m_2^2) \quad (\text{A.17})$$

$$a^2m^2 = m_1^2m_2^2 \quad (\text{A.18})$$

Using (A.15)-(A.18) leads to the values of

$$A = -B = \frac{1}{a\sqrt{a^2 + 4m^2}} \quad (\text{A.19})$$

$$m_1^2 = \frac{a^2}{2} \pm \frac{a}{2}\sqrt{a^2 + 4m^2} \quad (\text{A.20})$$

$$m_2^2 = \frac{a^2}{2} \mp \frac{a}{2}\sqrt{a^2 + 4m^2} \quad (\text{A.21})$$

and allows us to rewrite the propagator as

$$G(w) = \frac{\frac{1}{a\sqrt{a^2+4m^2}}}{w^2 - \left(\frac{a^2}{2} + \frac{a}{2}\sqrt{a^2 + 4m^2}\right)} - \frac{\frac{1}{a\sqrt{a^2+4m^2}}}{w^2 - \left(\frac{a^2}{2} - \frac{a}{2}\sqrt{a^2 + 4m^2}\right)} \quad (\text{A.22})$$

We have now transformed what we thought was a propagator for a single particle into that of two particles. The second term of the propagator should look troubling as it has a negative mass squared term and on top of that there is an overall negative in front of it. If the propagator was used to construct a Hamiltonian, we would see that the second term leads to a particle with a negative energy kinetic term or an Ostrogradsky instability. This negative energy means that there is no lower bound on the energy of the system, so any push from a metastable position could lead to a drastic drop of energy that is possibly infinite.

Appendix B

Perturbation Derivations

Only V_1

From (42) with $a = \text{constant}$, we see that there are no derivatives of V_1 . The components of the Stress-Energy Tensor that we need are then given by

$$\begin{aligned}\delta T_\varphi^t &= 2\dot{\phi}(t)\tilde{V}_1\partial_t\partial_\varphi\psi \\ \delta T_\varphi^r &= \frac{\tilde{V}_1}{a^2r}[-\partial_\varphi\xi + \partial_\varphi\psi + r\partial_r\partial_\varphi\xi - r\partial_r\partial_\varphi] \end{aligned} \tag{B.1}$$

The T_φ^t term can be rewritten as

$$0 = \partial_t\psi = \partial_\varphi\psi \tag{B.2}$$

which implies that ψ is a constant with respect to t and φ . We can then rewrite the T_φ^r as

$$0 = \frac{\tilde{V}_1}{a^2}r\partial_r\left[\frac{1}{r}(\xi - \psi)\right] \tag{B.3}$$

From this we see that $\xi = \psi$. The fact that ξ and ψ have no time dependence means

they don't actually vary in time. These perturbations no longer represent any degrees of freedom as they have no ability to vary in time. We have lost two degrees of freedom out of an expected three, so we can say that this is unphysical.

Only V_4

With only V_4 , (43) becomes

$$\tilde{V}'_4 = \frac{a}{\dot{\phi}} \sqrt{-2\rho(a)} \quad (\text{B.4})$$

Two things can be done here. One of them is to actually solve for \tilde{V}_4 as a function of time by treating \tilde{V}'_4 as $\frac{d\tilde{V}_4}{d\phi}$ and $\dot{\phi}$ as $\frac{d\phi}{dt}$ to get

$$\begin{aligned} \frac{d\tilde{V}_4}{dt} &= a \sqrt{-2\rho(a)} \\ \tilde{V}_4 &= a \sqrt{-2\rho(a)} t \end{aligned} \quad (\text{B.5})$$

The analytic form of \tilde{V}_4 in this case is nice, but serves no practical purpose here. From (B.4) we can also get an equation relating \tilde{V}'_4 and \tilde{V}''_4 . We get this by taking the derivative of \tilde{V}'_4 with respect to time.

$$\begin{aligned} \tilde{V}''_4 \dot{\phi} &= \frac{-a}{\dot{\phi}^2} \ddot{\phi} \sqrt{-2\rho(a)} = -\tilde{V}'_4 \frac{\ddot{\phi}}{\dot{\phi}} \\ \tilde{V}''_4 \dot{\phi}^2 + \tilde{V}'_4 \ddot{\phi} &= 0 \end{aligned} \quad (\text{B.6})$$

Using (B) along with components of the Stress-Energy Tensor T_φ^t , T_φ^r and T_r^r we can determine the behavior of the perturbations. From

$$\delta T_\varphi^r = -\frac{r(-1 + kr^2)\tilde{V}_2'}{a^2}[(\tilde{V}_2'\ddot{\phi} + \tilde{V}_2''\dot{\phi}^2)\partial_\varphi\partial_r\frac{\delta\phi}{r} + \frac{1}{2}\tilde{V}_4'\dot{\phi}^2\partial_\varphi\partial_r\frac{\xi - \psi}{r}] \quad (\text{B.7})$$

and (B) we are able to see that

$$\xi = -\psi \quad (\text{B.8})$$

Now we use

$$\delta T_\varphi^t = \frac{\tilde{V}_2'\dot{\phi}}{a^2}[k\partial_\varphi\delta\phi + a^2\dot{\phi}\partial_t\partial_\varphi\psi] \quad (\text{B.9})$$

to see that

$$\delta\phi = \frac{-a^2\dot{\phi}}{k}\dot{\psi} \quad (\text{B.10})$$

We now take our last two results and combine them with δT_r^r , which has become too big and cumbersome to just look at the equation to grasp, to get that

$$\delta\phi = \psi = \xi = 0 \quad (\text{B.11})$$

Only V_1 and V_4

The first thing we want to do is get an equation relating derivatives of \tilde{V}_1 to derivatives of \tilde{V}_4 from (21) to get

$$\tilde{V}_1' = \tilde{V}_4'\tilde{V}_4''\dot{\phi}^2 + (\tilde{V}_4')^2\ddot{\phi} \quad (\text{B.12})$$

or

$$\ddot{\phi} = \frac{\tilde{V}_1' - \tilde{V}_4'\tilde{V}_4''\dot{\phi}^2}{(\tilde{V}_4')^2} \quad (\text{B.13})$$

Using (B.13) after integrating by r and φ and δT_φ^r we can get an equation relating ψ and ξ .

$$\xi = \psi \frac{2\tilde{V}_1 + (\tilde{V}'_4)^2 \dot{\phi}^2}{2\tilde{V}_1 - (\tilde{V}'_4)^2 \dot{\phi}^2} \quad (\text{B.14})$$

and using δT_φ^t , after integrating with respect to φ , and its time derivative we can find equations relating $\delta\ddot{\phi}$ and $\delta\dot{\phi}$ to ψ .

Now we can use (B.14), the solutions for $\delta\ddot{\phi}$ and $\delta\dot{\phi}$, and T_r^r to find a differential equation describing ψ . After some simplifying the equation found is of the form

$$\dot{\psi} + (k^2 - m^2)\psi = 0 \quad (\text{B.15})$$

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