Case Western Reserve University

Senior Project Report

Rate of Convergence Calculations for Stochastic Integrals

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May 2, 2014

1 Abstract

Alternative definitions of the stochastic integral are investigated for their efficiency in numerical computation. Specifically, the Riemann-Stieltjes pathwise definition of the Itô and Stratonovich integrals is adapted to define integrals based on an arbitrary sampling point $\gamma \in [0, 1]$. Each definition is calculated numerically to determine the rate of convergence. Integrals are driven by the Brownian Motion (Wiener) process. Computationally advantageous definitions are found for particular integrands.

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2 Motivation

Stochastic models are an indispensable tool for creating realistic models of physical systems, inclusive of a system's inherent randomness. Ideally suited for describing physical diffusions, stochastic models also find extensive application in mathematical finance. The mathematical theory describing the construction and properties of these models has been extensively developed for models driven by the Brownian noise process. Brownian noise describes continuous diffusion processes with Gaussian-distributed increments, such as the diffusion of a drop of ink in still water.

Practical application of these models involves numerical approximation and computer simulation of solutions to stochastic differential equations (SDEs), thus efficient simulation methods are a priority for researchers. The focus of this project is on comparing different approaches to numerical simulation of SDEs driven by the Brownian Motion process. Specifically, we exploit a particular ambiguity in the definition of the stochastic integral, namely the choice of sample point for construction of Riemann sums. This choice can be continuously parametrized by $\gamma \in [0, 1]$, with each choice of γ yielding a distinct definition of the stochastic integral. The goal of this project is to write a *MATLAB* script that performs numerical integration for varying values of the parameter γ and compares the order of convergence of the integration scheme for each choice of γ.

It is the null hypothesis that all choices should converge equally quickly, but the discovery of a particular γ value with significantly higher order of convergence would bear great significance for numerical simulations. While the most computationally efficient choice of γ may not be the best choice for modeling reasons, we are likely able to convert a solution computed using one choice of γ to a solution corresponding to a different choice. There are known formulas[1] for converting between solutions corresponding to $\gamma = 0$ and $\gamma = 1/2$. Extending these results, one would then calculate solutions using a computationally advantageous definition of the stochastic integral and translate this solution to solve an SDE defined by the choice of γ prescribed by modeling choices. This project probes for such a computationally advantageous definition of the stochastic integral.

[1] Wong, Zakai (1965).

3 Introduction

We begin by introducing the stochastic process known as Brownian Motion, then elaborate on how the above-referenced parameter γ enters into the definition of the stochastic integral and explain how different values of γ lead to different definitions of the stochastic integral.

3.1 Brownian Motion

The Brownian Motion process can be introduced in many ways with varying levels of rigor. In this introduction, we shall sacrifice rigor for physical intuition and present Brownian Motion as the limit of a random walk.

A random walk in one dimension is the path of a particle that jumps forward or backward one unit at each discrete time step. Thus if the particle sits at position n at time t_i, then at time t_{i+1} the particle will be found either at $n+1$ or $n-1$ (see Figure 1).

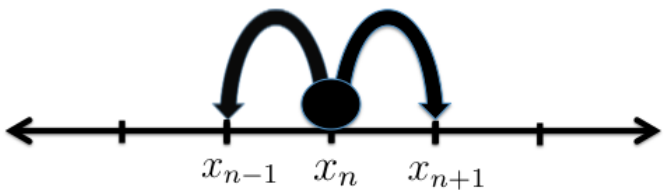

Figure 1: In a one-dimensional random walk, a particle jumps either forward or backward one discrete step at each discrete time point.

From this description, we can describe the probability of finding the random walk at a particular location recursively. Calling $p(t_i, x_n)$ the probability of finding the particle at the location x_n at time t_i , we have

$$p(t_i, x_n) = \frac{1}{2}p(t_{i-1}, x_{n-1}) + \frac{1}{2}p(t_{i-1}, x_{n+1}), \tag{1}$$

since the particle can only arrive at position n by jumping there from position x_{n-1} or position x_{n+1} (note that we are assuming that the particle jumps forwards or backwards with equal probability and that at $t_0 = 0$ the particle starts at the origin, $p(0, x_n) = \delta_0(x_n)$). We now employ a clever

trick: subtract from both sides of (1) the probability that the particle could be found at position x_n at time t_{i-1}:

$$\begin{aligned} p(t_i, x_n) - p(t_{i-1}, x_n) &= \frac{1}{2}\left[p(t_{i-1}, x_{n-1}) + p(t_{i-1}, x_{n+1}) - 2p(t_{i-1}, x_n)\right] \\ &= \frac{1}{2}\left[p(t_{i-1}, x_{n+1}) - p(t_{i-1}, x_n)\right] - \frac{1}{2}\left[p(t_{i-1}, x_n) - p(t_{i-1}, x_{n-1})\right]. \end{aligned}$$

Now we begin to see a difference equation emerging: dividing by Δt on the left-hand side and $(\Delta n)^2$ on the right-hand side, we have

$$\frac{p(t_i, x_n) - p(t_{i-1}, x_n)}{\Delta t} = \frac{1}{2}\frac{\left[p(t_{i-1}, x_{n+1}) - p(t_{i-1}, x_n)\right] - \left[p(t_{i-1}, x_n) - p(t_{i-1}, x_{n-1})\right]}{(\Delta n)^2}$$

so that the left-hand side is a first-difference with respect to time and the right-hand side is a second difference with respect to the spatial variable. In a continuum limit $\Delta t \to 0$, $\Delta n \to 0$ we have a partial differential equation [2]:

$$\frac{\partial p}{\partial t}(t, x) = \frac{1}{2}\frac{\partial^2 p}{\partial x^2}(t, x). \tag{2}$$

That is, the probability distribution function of the Brownian motion process satisfies the heat equation. If we assume that the particle starts at the origin with probability 1, then the initial state of the particle is a Dirac delta measure, δ_0. Evolving this distribution with the heat kernel, we have

$$p(t, x) = \frac{1}{(2\pi t)^{1/2}} e^{-x^2/2t}. \tag{3}$$

Thus the probability distribution function for Brownian Motion starting at the origin is a Gaussian distribution with mean 0 and standard deviation $\sqrt{t}$. A few sample paths are plotted in Figure 2 along with the curve $y = \sqrt{t}$ to illustrate this property.

3.2 Stochastic Differential Equations

A key feature of the Brownian Motion process is that its paths are (almost surely) nowhere-differentiable. This makes it difficult to incorporate

[2] This limit requires that we treat time and space unequally: since we divided the left-hand side of (1) by Δt and the right-hand side by $(\Delta n)^2$, we must scale parabolically in this limit, *i.e.* such that we maintain $\Delta t/(\Delta n)^2 = 1$

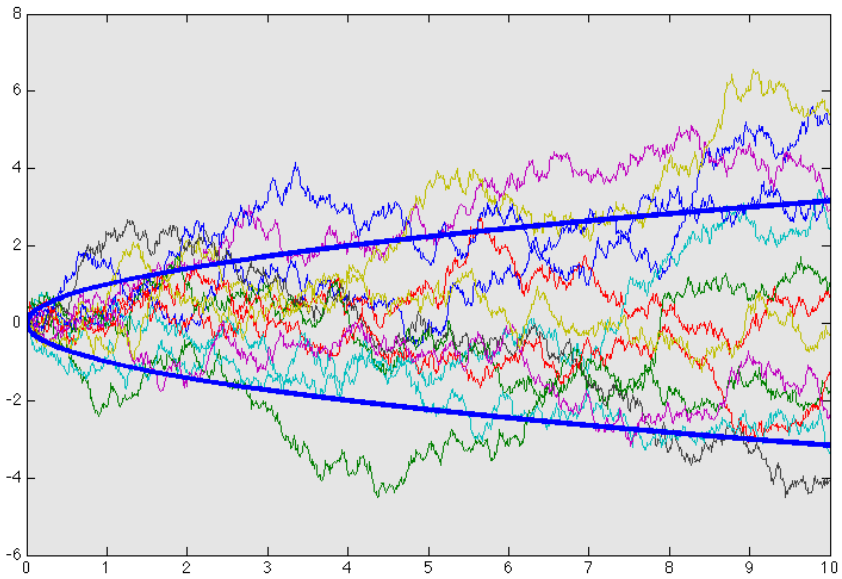

Figure 2: Sample paths of the Brownian Motion process on the interval $[0, 10]$. At time t, the process is Gaussian distributed with mean 0 and standard deviation $\sqrt{t}$ (plotted as thick line).

the Brownian process into a differential equation model, since we would inevitably start manipulating derivatives that don't exist. To avoid this difficulty, we always reinterpret a stochastic differential equation as an integral equation. With a suitable definition of the integral, these models can be handled rigorously to describe the dynamics of a system subject to randomly evolving forces. In this paper, we deal only with first-order linear stochastic differential equations driven by the Brownian Motion process. The simple population growth model is an example.

In what follows, we will again sacrifice some rigor in order to better develop our intuition. The simplest version of the population growth model assumes that the instantaneous population growth is proportional to the current population size, here denoted X_t:

$$\frac{dX_t}{dt} = \alpha X_t,$$

where α is a constant determined by the birth and death rates of the population.

In order to improve this model, we will assume that there are uncorrelated "noise" corrections to the proportionality constant α. In a discrete model, increments of the random walk would work perfectly well. So in this continuous model, we will naively use the derivative of Brownian Motion for our noise:

$$\begin{aligned}\frac{dX_t}{dt} &= \left(\alpha + \text{“}\frac{dB_t}{dt}\text{”}\right) X_t \\ &= \alpha X_t + \text{“}\frac{dB_t}{dt}\text{”} X_t.\end{aligned}$$

Of course, the derivative $\frac{dB_t}{dt}$ does not actually exist, so, formally, we multiply through by dt

$$dX_t = \alpha X_t dt + X_t dB_t \tag{4}$$

and reinterpret as an integral equation

$$\int_0^T dX_t = \int_0^T \alpha X_t dt + \int_0^T X_t dB_t$$

or

$$X_T = X_0 + \int_0^T \alpha X_t dt + \int_0^T X_t dB_t. \tag{5}$$

Once we have a suitable interpretation for $\int_0^T X_t dB_t$ then we will have constructed a stochastic population growth model[3].

3.3 Riemann-Stieltjes Integrals

The central concept in this research will be the stochastic integral. As demonstrated above, we interpret a stochastic differential equation (4) as an integral equation (5). But what does the term $\int_0^T X_t dB_t$ in (5) mean? Norbert Wiener defined this integral rigorously by constructing an appropriate measure on the set of continuous functions on an interval. For the sake of numerical integration, however, it is convenient to define the stochastic integral as a Riemann-Stieltjes integral on individual paths of the Brownian process.

[3]This example follows Øksendal, *Stochastic Differential Equations*, 2010.

Recall the Riemann definition of a definite integral, where we partition the domain of our function, pick sample points t_i^* in each interval of the partition, evaluate our function at each sample point, and use these values to construct the Riemann sum. Taking the limit of our Riemann sum as the size of intervals in the partition approaches zero yields the value of the Riemann integral–*independent of what sample points we choose to evaluate our function.*

$$\int_0^T f(t)dt = \lim_{\Delta t \to 0} \sum_i f(t_i^*)\Delta t_i.$$

We analogously define the Riemann-Stieltjes integral, where a function f is integrated against another function g: $\int_0^T f(t)dg(t)$. If our function g is differentiable, we have the familiar

$$\int_0^T f(t)dg(t) = \int_0^T f(t)\frac{dg}{dt}(t)dt.$$

To allow for non-differentiable functions g, we define the Riemann-Stieltjes integral as the limit of a Riemann sum:

$$\int_0^T f(t)dg(t) = \lim_{\Delta t \to 0} \sum_i f(t_i^*)\left[g(t_{i+1}) - g(t_i)\right]. \tag{6}$$

This Riemann-Stieltjes integral of f against g can be thought of as an integral where the values of f are weighted by the rate at which the values of g are changing at each point. If g is smooth enough, we again do not need to worry about the sample point t_i^* where we evaluate f in each interval.

In the case of stochastic integrals, where the function g that we integrate against is a Lévy process representing our noise, we will find that this choice of sample point has significant consequences for the result of the integral. This is because the paths of Brownian Motion aren't smooth; in fact, they are almost surely nowhere differentiable.

This means that each possible way of choosing a sample point t_i^* from each interval leads to a *different* definition of the stochastic integral. To keep track of the possible choices of sampling point, we introduce the parameter $\gamma \in [0, 1]$ as the relative position of the chosen sample point within each interval:

$$t_i^* = (1 - \gamma)t_i + \gamma t_{i+1}$$

For example, choosing $\gamma = 0$ corresponds to defining the stochastic integral by sampling each interval at the left-endpoint (*i.e.* the Itô integral); $\gamma = 1/2$ corresponds to the midpoint (Stratonovich integral); $\gamma = 1$ to the right-endpoint.

3.4 Itô and Stratonovich Definitions

In practice, however, we only commonly use two definitions of the stochastic integral: the Itô and Stratonovich integrals. The Itô integral is constructed by evaluating the function f at the left-endpoint t_i of each interval ($\gamma = 0$). The Stratonovich integral is constructed by evaluating f at the midpoint $\bar{t}_i = \frac{t_i + t_{i+1}}{2}$ of each interval ($\gamma = 1/2$). Each choice possesses a particularly nice property when the integral is considered with respect to the Brownian motion process: when we use Itô's definition, the indefinite integral obtained can be shown to possess the martingale property; when we use Stratonovich's definition, indefinite integrals can be calculated using the familiar calculus rules for anti-differentiation.

As indicated above, there are formulas for converting between solutions to the Itô and Stratonovich interpretations of the stochastic integral (due to Wong and Zakai for the stochastic integral driven by Brownian motion[4] and extended by Kurtz, Pardoux, and Protter to stochastic integrals driven by a particular class of semi-martingales[5]). We state this formula here for the case of stochastic integrals driven by Brownian motion: denoting the Itô integral of a function $f(t, B_t)$ depending on time and the value of a Brownian motion process by $\int f(t, B_t) dB_t$ and the Stratonovich integral by $\int f(t, B_t) \circ dB_t$, we have[6]

$$\int_0^T f(t, B_t) \circ dB_t = \frac{1}{2} \int_0^T \frac{\partial f}{\partial B_t}(t, B_t) dt + \int_0^T f(t, B_t) dB_t.$$

Noting that the far-left and far-right terms in this equation are simply the Stratonovich and Itô integrals, respectively, we see that these two differ simply by the middle term, which is an ordinary integral whose integrand is the partial derivative of the function f with respect to the second slot.

Kurtz *et al.* extend this result to integrals driven by a Lévy process, though only for a particular class of integrands. While these formulas hold

[4] Wong, Zakai (1965).

[5] Kurtz, Pardoux, Protter (1992).

[6] Øksendal, *Stochastic Differential Equations*, 2010.

only for a limited class of integrands and (so far as the author has been able to find) only to convert between the $\gamma = 0$ and $\gamma = 1/2$ definitions of the stochastic integral, finding a γ value particularly well-suited to numerical calculation would spur great interest in developing formulas to convert between definitions of the stochastic integral for all $\gamma \in [0, 1]$.

4 Literature Review

The theory of stochastic differential equations driven by the Brownian motion process has been extensively studied and is described in a number of textbooks, such as Bernt Øksendal's *Stochastic Differential Equations* (2010). A standard handbook for numerical approximation and simulation of SDEs driven by the Brownian process is Peter Kloeden and Eckhard Platen's *Numerical Solution of Stochastic Differential Equations* (1992), which discusses both approximation algorithms and order of convergence computations.

The motivation to move beyond the Brownian motion process and consider Lévy processes is described in the article Lévy Processes in the Physical Sciences (2001) by project advisor Wojbor Woyczyński. In this article, Woyczyński offers four examples of how Lévy α-stable distributions arise from first principles in physical systems. The examples offered show how hitting times of a Brownian motion particle follow a Lévy distribution ($\alpha = 1/2$), how particles emitted from a point source follow a Cauchy distribution ($\alpha = 1$) in the detection plane, and how the Holtsmark distribution ($\alpha = 3/2$) arises to describe the gravitational field generated by stars distributed uniformly in space and to describe the size distribution of large polymerized molecules. Further discussion of α-stable distributions arising from first-principles arguments is found in Zolotarev's article One-Dimensional Stable Distributions (1986).

Standard reference on the properties of Lévy processes is Jean Bertoin's *Lévy Processes* (1996). David Appelbaum's *Lévy Processes and Stochastic Calculus* (2009) approaches the subject with a more physics-oriented mindset and offers a more intuitive discussion of the Lévy process and stochastic integrals driven by Lévy processes. A full development of the far more general theory of integration with respect to an arbitrary semi-martingale is found in Stanisław Kwapień and Wojbor Woyczyński's *Random Series and Stochastic Integrals* (2002).

An early survey of numerical treatment of Lévy processes is Aleksander

Janicki and Aleksander Weron's article Can One See α-Stable Variables and Processes? (1994). This paper discusses extending some of the numerical techniques outlined by Kloeden and Platen for processes driven by Brownian motion to be applied to processes driven by Lévy processes.

The original result for converting between Itô and Stratonovich definitions of the stochastic integral driven by Brownian motion is Eugene Wong and Moshe Zakai's article On the Convergence of Ordinary Integrals to Stochastic Integrals (1965). Thomas Kurtz, Etienne Pardoux, and Philip Protter extend this result to apply to stochastic integrals driven by Lévy processes in their article Stratonovich Stochastic Differential Equations Driven by General Semimartingales (1992).

5 Objectives and Methods

This project explores the alternative definitions of the stochastic integral from the point of view of numerical computation. Because practical implementation of stochastic models always involves computer simulation, efficient numerical computation of the stochastic integral is of great importance. Thus the objective of this project is to probe for a definition that lends itself particularly well to numerical computation. That is, a choice of $\gamma \in [0, 1]$ for which the rate of convergence of the numerical approximation is highest.

We probe using a *MATLAB* script where one chooses an integrand and the script performs numerical calculations of the stochastic integral for multiple values of γ. The script computes rate of convergence statistics for each chosen definition of the stochastic integral and compares these.

5.1 Explanation of *MATLAB* Script

Figure 3 presents a schematic overview of the *MATLAB* script.

The user provides two pieces of input: the desired integrand (a function of time and the Brownian Motion process) and the mesh fineness, *i.e.* how finely to discretize the interval $[0, T]$ (in this project, we always chose $T = 10$).

Given this mesh fineness, a script produces 100 paths of the Brownian process on the corresponding discretization of $[0, T]$. Paths of the process are simulated as follows: from the above form of the probability distribution function (3) of Brownian Motion along with the Markov property, we see that the increments of Brownian Motion of width Δt are Gaussian distributed

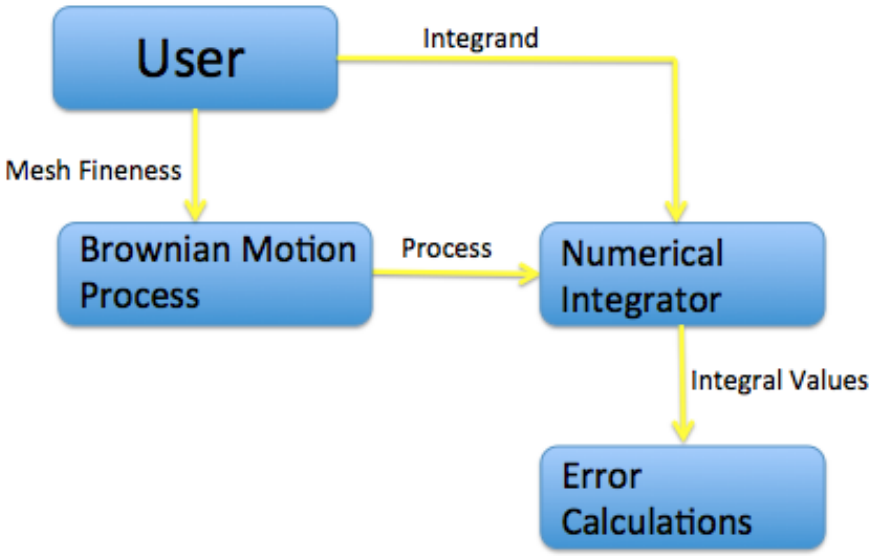

Figure 3: Schematic overview of *MATLAB* script. The user supplies both the integrand and mesh fineness. Paths of the Brownian Motion process are generated on this mesh and inputted to the numerical integrator, which calculates the Riemann-Stieltjes sum in (6). These values of the integral are then inputted to a script that computes the approximation error.

with mean 0 and standard deviation $\sqrt{\Delta t}$. So to simulate a path of Brownian Motion over $[0, T]$ broken into 100 time points, we

1. Generate 100 random numbers, $\mathcal{N}\left(0, \sqrt{T/100}\right)$ distributed.
2. For each $t_i = i\frac{T}{100}$, compute the cumulative sum of the first i normally distributed random numbers.
3. Set B_{t_i} equal to the above cumulative sum.

Based on the user's chosen integrand, the numerical integrator script then computes the integrand using the previously generated Brownian path. For each path, the script computes the Riemenn-Stieltjes sum in (6). The collection of these values for all simulated Brownian paths is the numerical approximation to the stochastic integral.

Finally, a third script computes the error in numerical approximation. The error is calculated as mean-squared error. That is, for each simulated path of the Brownian process, we calculate the square of the difference between the "actual" value of the integral and the approximated value, then average over all simulated paths:

$$\text{Error} = \sum \frac{(\text{Actual} - \text{Approximate})^2}{\text{Number of Paths}}.$$

Of course, an exact "actual" value is not always available, just as closed-form expressions of ordinary integrals do not always exist. So to make this script applicable to an arbitrary integrand, the "actual" value of the integral is taken to be the numerical approximation calculated on the finest mesh that the computer can handle in a reasonable amount of time (usually 10,000 time points).

6 Results and Analysis

For a chosen integrand, say $\int_0^{10} B_t dB_t$, we run the above *MATLAB* script and gather a plot of approximation error versus mesh fineness. These are best represented with logarithmically scaled axes. See Figure 4 for examples of these plots for nine definitions of the integral $\int_0^{10} B_t dB_t$.

When we are lucky, the approximation error decreases linearly with mesh fineness on a log-log scale, indicating that the actual relationship is exponential decay. In these cases, we can fit a line to the approximation error versus mesh fineness curves to compare their steepness. It is this steepness that indicates which definition presents the greatest computational advantage, since this indicates how quickly the approximation error dies off as we increase the mesh fineness.

For easy comparison, we extract the slope of each best fit line and plot these as a function of $\gamma \in [0, 1]$ (see Figure 5). Since the slopes of these lines are negative, the lowest value corresponds to the steepest slope and thus to the best definition for computation with that particular integrand. Figure 5 indicates that the Itô definition of the integral $\int_0^{10} B_t dB_t$ is the best for computation, since the line of best fit for the approximation error versus mesh fineness curve has the steepest slope.

In some sense, this provides a positive answer to our original question: is there a choice of $\gamma \in [0, 1]$ for which numerical approximations of the stochas-

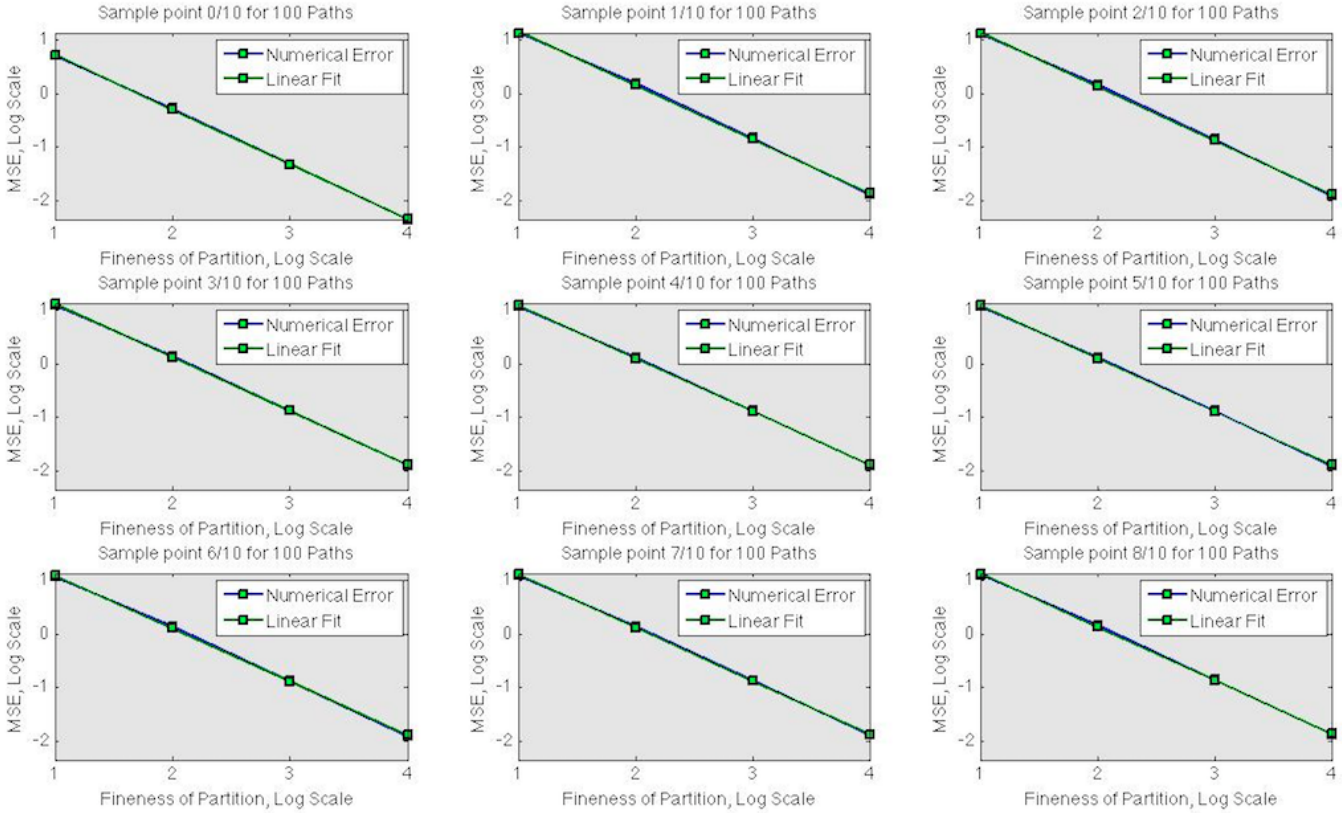

Figure 4: Approximation error versus mesh fineness for nine definitions of the stochastic integral $\int_0^{10} B_t dB_t$. Shown are the actual data (blue) and linear fits (green). In most cases the linear fit lies directly on top of the data. This linear relationship on a log-log plot implies that the approximation error decays exponentially with increasing mesh fineness.

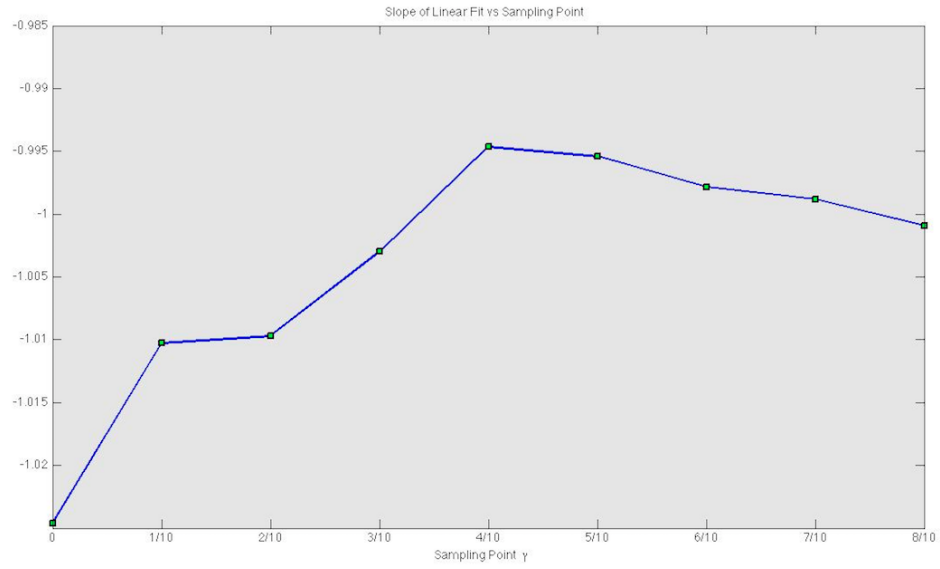

Figure 5: Slope of best fit line to approximation error versus mesh fineness curve plotted against chosen definition of the stochastic integral $\int_0^{10} B_t dB_t$, *i.e.* choice of $\gamma \in [0,1]$. Note that $\gamma = 0$, the Itô integral, has the steepest slope, indicating that for this particular integrand, the Itô integral is the most computationally efficient definition.

tic integral converge more quickly than for other values of γ? However, it is not an entirely satisfying answer. The plots in Figures 4 and 5 indicate an advantage for the Itô integral, but only for one particular integral: $\int_0^{10} B_t dB_t$.

The most satisfying answer we could hope for would be the discovery of some $\gamma \in [0,1]$ that provides a computational advantage for numerical approximations with all integrands. However, the same analysis performed in Figures 4 and 5 for an exponential integrand, $\int_0^{10} \exp(B_t)\, dB_t$, indicates that this is not the case. In Figure 6 we have approximation error versus mesh fineness plots for the exponential integrand. These do not follow a linear trend as well as the same plots for the linear integrand, but they are close enough that the linear fit is reasonable. Figure 7 shows the slopes of the best fit lines plotted against the sampling point γ. What we observe is that the slope is now steepest for the right-endpoint definition of the integral–the exact opposite of what we found for the linear integrand.

Thus we cannot choose a sampling point that is most efficient for compu-

tation of an arbitrary stochastic integral, since we have seen that the linear and exponential integrands are most efficiently approximated by different choices of sampling point. The next best thing to hope for is to find a choice of sampling point that is most efficient for computation with a particular class of integrands, *e.g.* polynomial integrands. While examining the polynomial integrands, however, we encounter a problem. Beyond a certain degree, the approximation error versus mesh fineness plots can no longer be approximated by a linear fit on a log-log plot. Figure 8 illustrates this problem in the case where the integrand is the 20^{th} power of Brownian Motion. In each case there is a much greater improvement in the error when increasing the mesh fineness from 10^2 time points to 10^3 time points than is achieved when increasing the mesh fineness from 10 to 10^2 time points. Thus we cannot apply the linear fit to compare the steepness of these curves. In the absence of such a simple comparison, it is unclear how to systematically draw conclusions about the computational efficiency of each choice of sampling point.

7 Conclusion

In this project we have successfully written a *MATLAB* script for numerical approximation of alternative definitions of the stochastic integrals defined by choosing an arbitrary sampling point $\gamma \in [0, 1]$ for the Riemann-Stieltjes sum. This script tracks the approximation error of the calculation for a given mesh fineness and compares the rate at which this error decreases with increasing mesh fineness for different choices of sampling point.

The objective of the project was to explore whether there is a particular choice of sampling point that would offer the best improvement in approximation error as we increase mesh fineness. We demonstrate that such a choice often exists for a particular stochastic integral (the examples presented in this report are the linear and exponential integrands, $\int_0^{10} B_t dB_t$ and $\int_0^{10} \exp(B_t)\, dB_t$). However, the results are inconsistent in the sense that the choice of γ that is best for one integrand will not be the best choice for all integrands, as the linear and exponential integrand examples illustrate.

We hope to find a class of integrands for which there is a single most efficient choice of sampling point. An obvious first choice to examine would be the polynomial integrands, *i.e.* finite sums of positive integer powers of the Brownian Motion process. However, we have not yet successfully

analyzed the polynomial integrands because for high degree polynomials, the approximation error does not decrease linearly with the mesh fineness on a log-log scale (see Figure 8). This does not preclude an analysis of polynomial integrands on an individual basis, but it does complicate the task of finding a systematic comparison of these integrands. A continuation of this work will involve devising a comparison of the rates of convergence that does not rely on using linear fits.

Another way to extend this work would be to introduce another measure of computational efficiency, namely computation time. In this project, computational efficiency was measured only by the approximation error. Of course, in applications where many of these computations must be performed quickly (such as mathematical financial analysis), computation time is a great concern. If one were to introduce this measure of efficiency, then it is likely that the gains in accuracy from using a non-standard sampling point are outweighed by the computational cost of simulating the Brownian Motion process on the time points between those used to calculate the increments of Brownian Motion—in this case Itô's integral would save us from performing these computations.

Finally, this project should be extended to integrals driven by more general processes than the Brownian Motion process, such as α-stable Lévy processes. These processes allow for models that incorporate random, discontinuous jumps and are therefore very popular for modeling. Their calculus is even more subtle than stochastic calculus for stochastic differential equations driven by Brownian Motion, so good numerical simulation is again essential for stochastic integrals driven by Lévy processes. We are attempting to apply the script used in this project to stochastic integrals of paths of a Lévy process, but have not developed any results so far.

This project was originally conceived as an exploration of the Itô and Stratonovich definitions of the integral. When I first approached my advisor, Professor Woyczyński, I wanted to understand why these two formulations gave rise to different results. Moreover, I wanted to know which one was "right." As might be expected, such an ill-posed question has no answer; there are settings where there are convincing arguments for using Itô's formulation and settings with equally convincing arguments for Stratonovich's interpretation.

So while I did not answer the almost philosophical question of Itô versus Stratonovich, I did learn the value of computation. I was hesitant at first to dirty my hands with "for" loops and random number generation, but I found

that forcing myself to perform computations enhanced my understanding. As I move forward into the study of more general stochastic processes, I now have the tools to take the abstract and make it tangible. In a broad sense, the study of stochastic processes is looking for patterns in the randomness of our physical world in order to tame it. In simulating these processes, however, we unpack their neat description and, with the push of a button, relinquish our control over randomness.

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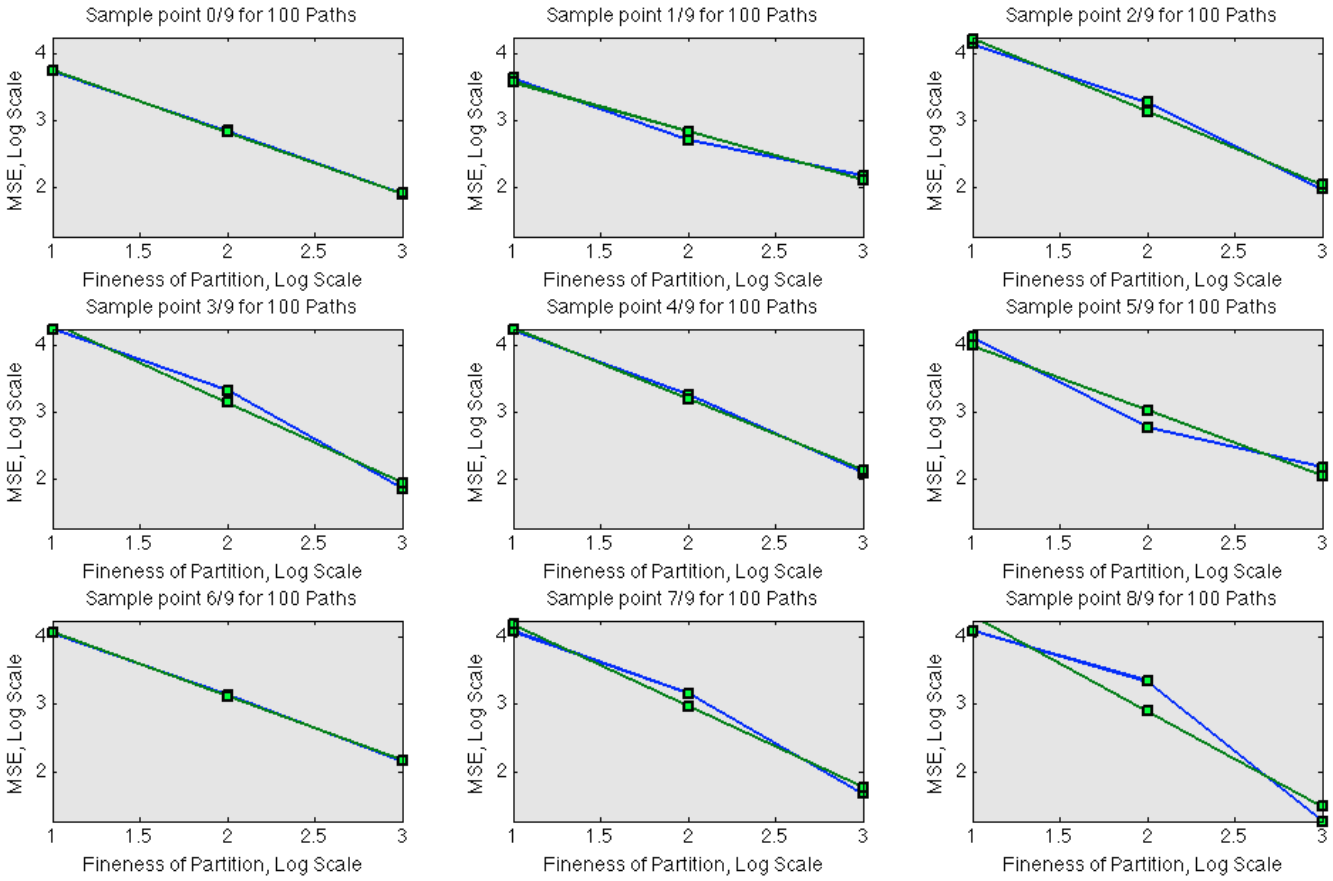

Figure 6: Approximation error versus mesh fineness for nine definitions of the stochastic integral $\int_0^{10} \exp(B_t)\, dB_t$. Shown are the actual data (blue) and linear fits (green). For most definitions shown, the linear fit is a good approximation.

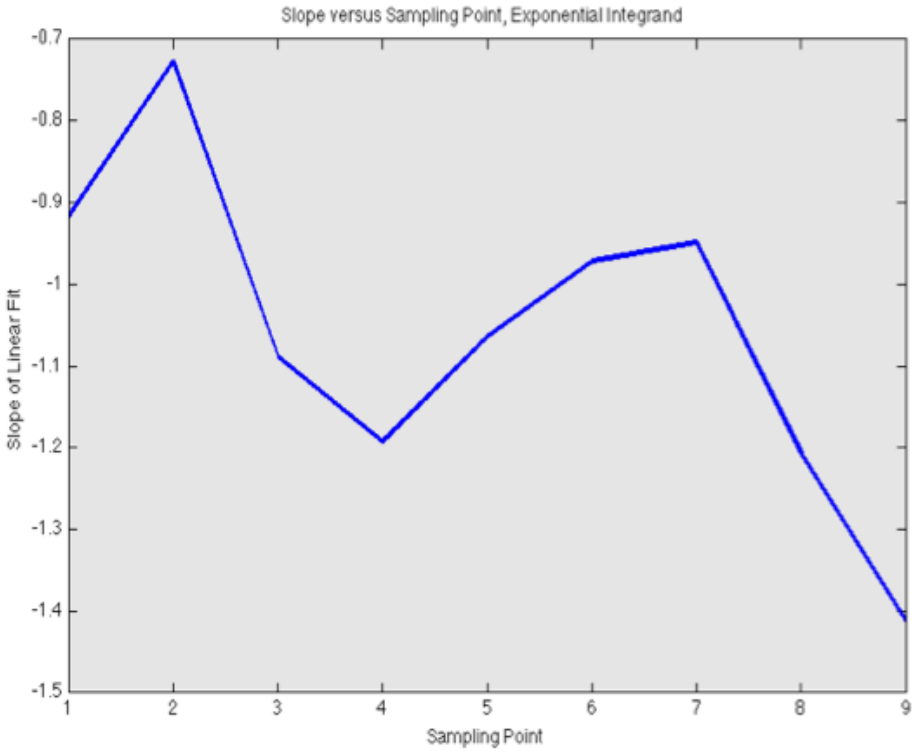

Figure 7: Slope of best fit line to approximation error versus mesh fineness curve plotted against chosen definition of the stochastic integral $\int_0^{10} \exp(B_t)\, dB_t$, *i.e.* choice of $\gamma \in [0,1]$. Whereas for a linear integrand, $\int_0^{10} B_t dB_t$, the Itô definition was best (cf. Figure 5), we see that in this case the right-endpoint definition appears to be the most efficient.

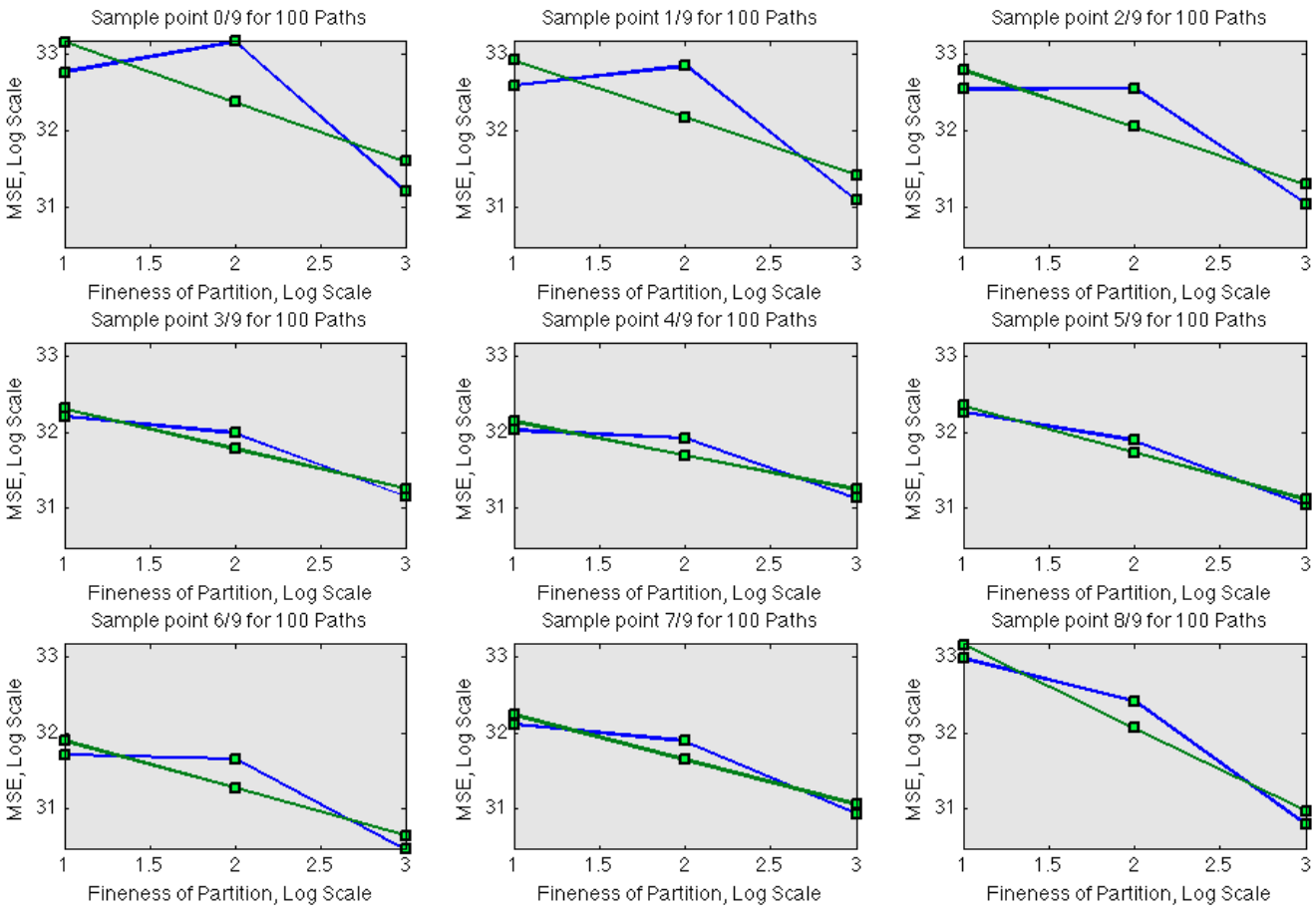

Figure 8: Approximation error versus mesh fineness for nine definitions of the stochastic integral $\int_0^{10} \exp(B_t)\, dB_t$. Shown are the actual data (blue) and linear fits (green). For most definitions shown, the linear fit is a good approximation.