Quiz Book Number

Math 121 Final

December 10, 2024

EF:

1-2	/20
3-4	/20
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9-10	/20
11-12	/20
13-14	/20
15-16	/20
17-18	/20
19-20	/20
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Name KEY

Directions:

1. No book, notes, or attaching chains to monkeys. You may use a calculator to do routine arithmetic computations. You may *not* use your calculator to store notes or formulas. You may not share a calculator with anyone.
2. You should show your work, and explain how you arrived at your answers. A correct answer with no work shown (except on problems which are completely trivial) will receive no credit. If you are not sure whether you have written enough, please ask.
3. You may not make more than one attempt at a problem. If you make several attempts, you must indicate which one you want counted, or you will be penalized.
4. You may leave as soon as you are finished, but once you leave the exam, you may not make any changes to your exam.

Have a Safe and Happy Break

1. (10 points)

(a) If you write $|2x-3|<2$ in the form $a<x<b$, what are a and b?

$-2 < 2x - 3 < 2$

$1 < 2x < 5$

$\frac{1}{2} < x < \frac{5}{2}$

$a = \frac{1}{2} \quad b = \frac{5}{2}$

(b) Solve for x, $e^{x^2} = e^{4x}e^{-3}$

$x^2 = 4x - 3$

$x^2 - 4x + 3 = 0$

$(x-1)(x-3) = 0$

$x = 1, 3$

2. (10 points)

(a) Find the inverse of the function $f(x) = \dfrac{x+3}{5x-3}$

$x = \dfrac{y+3}{5y-3}$

$(5y-3)x = y + 3$

$5xy - 3x = y + 3$

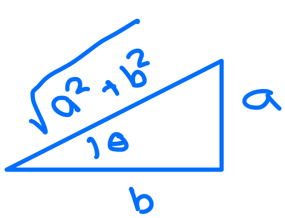
$5xy - y = 3x + 3$

$y = \dfrac{3x+3}{5x-1}$

$f^{-1}(x) = \dfrac{3x+3}{5x-1}$

(b) If $\tan\theta = \frac{a}{b}$ and $0 \le \theta \le \frac{\pi}{2}$ find

$$\sin\theta + \cos\theta$$



$\sin\theta = \dfrac{a}{\sqrt{a^2+b^2}} \quad \cos\theta = \dfrac{b}{\sqrt{a^2+b^2}}$

$\sin\theta + \cos\theta = \dfrac{a+b}{\sqrt{a^2+b^2}}$

3. (10 points) If $f(2) = 3$ and $f'(2) = -1$, find:

(a) $h'(2)$ where $h(x) = \dfrac{f(x)}{x}$

$$h'(x) = \frac{x\, f'(x) - f(x)}{x^2}$$

$$h'(2) = \frac{2(-1) - 3}{4} = -\frac{5}{4}$$

(b) $\displaystyle\lim_{x\to 0} \frac{f(2+x) - f(2)}{x}$

$$= f'(2) = -1$$

4. Find $f'(x)$ for each of the following:

(a) $f(x) = \cos^4 5x$

$$f'(x) = 4\left(\cos^3(5x)\right)\left(-\sin(5x)\right) \cdot 5$$

(b) $f(x) = \dfrac{e^{2x}}{2x + \ln x}$

$$f'(x) = \frac{(2x+\ln x)(2e^{2x}) - (e^{2x})\left(2 + \frac{1}{x}\right)}{(2x + \ln x)^2}$$

5. (10 points) Find $f'(x)$ for

(a) $f(x) = \ln(\arcsin x) + \cosh(\tan x)$

$$f'(x) = \frac{1}{\arcsin x} \cdot \frac{1}{\sqrt{1-x^2}} + \sinh(\tan x) \cdot \sec^2 x$$

(b) $f(x) = (\sin x)^x$ $\quad \ln f(x) = x \ln(\sin x)$

$$\frac{1}{f(x)} \cdot f'(x) = x \frac{\cos x}{\sin x} + \ln(\sin x)$$

$$f'(x) = (\sin x)^x \left[x \frac{\cos x}{\sin x} + \ln(\sin x) \right]$$

6. (10 points) For

$$x^2 + 2x + 4y^2 = 4 + y$$

(a) Find $\frac{dy}{dx}$

$$2x + 2 + 8y \frac{dy}{dx} = \frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{-2x-2}{8y-1}$$

(b) Find the points on the curve where the tangent line is horizontal.

$-2x - 2 = 0 \qquad x = -1$

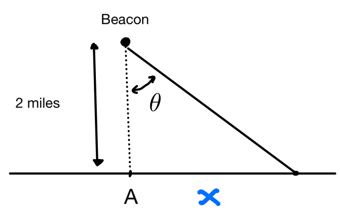
$1 - 2 + 4y^2 = 4 + y \qquad (-1, \frac{5}{4})$

$4y^2 - y - 5 = 0 \qquad (-1, -1)$

$(4y - 5)(y + 1) = 0$

$y = \frac{5}{4}, -1$

7. (10 points) A rotating beacon is located 2 miles out in the water. Let A be the point on the shore that is closest to the beacon. As the beacon rotates at 10 rev/min, the beam of light sweeps down the shore once each time it revolves. Assume that the shore is straight. How fast is the point where the beam hits the shore moving at an instant when the beam is lighting up a point 2 miles along the shore from the point A?



$$\tan\theta = \frac{x}{2}$$

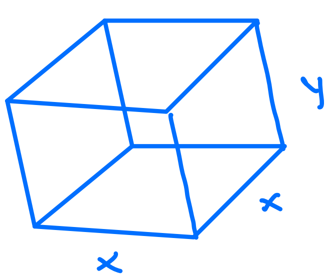
$$(\sec^2\theta)\frac{d\theta}{dt} = \frac{1}{2}\frac{dx}{dt}$$

$$\frac{d\theta}{dt} = \frac{10\,\cancel{\text{REV}}}{\text{MIN}} \cdot \frac{2\pi}{\cancel{\text{REV}}} = \frac{20\pi}{\text{MIN}}$$

$$x = 2 \Rightarrow \theta = \frac{\pi}{4} \Rightarrow \sec^2\left(\frac{\pi}{4}\right) = 2$$

$$2(20\pi) = \frac{1}{2}\frac{dx}{dt} \qquad \frac{dx}{dt} = 80\pi \text{ MILES}/\text{MIN}$$

8. (10 points) Jaimie wishes to build a rectangular tank for her pet snail. If the tank has a square base, an open top, and a volume of 108 cubic yards, find the dimensions of the tank that has the smallest surface area (no top).



$$x^2 y = 108 \qquad y = \frac{108}{x^2}$$

$$SA = x^2 + 4xy$$

$$SA = x^2 + \frac{432}{x}$$

$$\frac{dSA}{dx} = 2x - \frac{432}{x^2} = 0 \qquad 2x^3 = 432$$

$$x^3 = 216 \qquad x = 6 \quad y = 3$$

$$\frac{d^2SA}{dx^2} = 2 + \frac{864}{x}\Big|_{x=6} > 0 \qquad \text{MIN}$$

9. (10 points)

(a) Compute $\lim_{x \to 0} \frac{e^{4x} - e^{x}}{x}$ $= \text{"}\frac{0}{0}\text{"}$

$$\lim_{x \to 0} \frac{4e^{4x} - e^{x}}{1} = \frac{3}{1} = 3$$

(b) Compute $\lim_{x \to \pi/4} (1 - \tan x) \sec 2x$

$$= \lim_{x \to \frac{\pi}{4}} \frac{1 - \tan x}{\cos 2x} = \frac{0}{0} \quad \lim_{x \to \frac{\pi}{4}} \frac{-\sec^2 x}{-2\sin 2x}$$

$$= \frac{-2}{-2} = 1$$

10. (10 points)

(a) Using Newton's method, we know $x_0 = 2, x_1 = 4$ and $f(2) = 5$, what is $f'(2)$?

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} \Rightarrow 4 = 2 - \frac{5}{f'(2)}$$

$$f'(2) = -2.5$$

(b) Use a linear approximation to estimate $f(0.1)$, if $f(0) = 5$ and $f'(x) = \frac{\cos x}{x^2 + 2}$

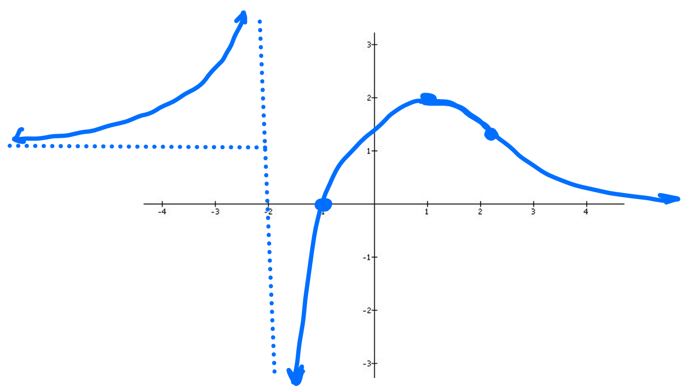
$$L(x) = f(a) + f'(a)(x - a)$$

$$L(x) = 5 + \frac{1}{2}x$$

$$f(0.1) \approx L(0.1) = 5 + (.5)(0.1) = 5.05$$

11. (10 points) Sketch the graph of a function $f(x)$ that satisfies ALL of the following:

- $f(-1) = 0$ and $f'(1) = 0$
- $f'(x) > 0$ when $x < -2$ and $-2 < x < 1$
- $f'(x) < 0$ when $x > 1$
- $f''(x) > 0$ when $x < -2$ and $x > 2$
- $f''(x) < 0$ when $-2 < x < 2$
- $\lim_{x\to-\infty} f(x) = 1$ and $\lim_{x\to+\infty} f(x) = 0$
- $\lim_{x\to-2^-} f(x) = +\infty$ and $\lim_{x\to-2^+} f(x) = -\infty$



12. (10 points)

(a) Compute: $\int (x + 3x^2)\, dx$

$$= \frac{x^2}{2} + x^3 + C$$

(b) Compute $\int \left(\frac{1}{x} + e^{7x} + \sec^2 x + \sin x\right)\, dx$

$$= \ln|x| + \frac{e^{7x}}{7} + \tan x - \cos x + C$$

13. (10 points)

(a) Compute: $\int \frac{1}{\sqrt{x}(1+\sqrt{x})}\,dx.$

$u = 1 + \sqrt{x}$

$du = \frac{1}{2\sqrt{x}}\,dx \quad dx = 2\sqrt{x}\,du$

$$= \int \frac{2\cancel{\sqrt{x}}}{\cancel{\sqrt{x}}\,u}\,du = 2\ln|u| = 2\ln|1+\sqrt{x}| + C$$

(b) Compute: $\int_0^1 \sqrt{x}(x^2+1)^2\,dx.$

$$= \int \sqrt{x}\,(x^4 + 2x^2 + 1)\,dx = \int (x^{\frac{9}{2}} + 2x^{\frac{5}{2}} + x^{\frac{1}{2}})\,dx$$

$$= \frac{2x^{\frac{11}{2}}}{11} + \frac{4}{7}x^{\frac{7}{2}} + \frac{2}{3}x^{\frac{3}{2}}\Big|_0^1 = \frac{2}{11} + \frac{4}{7} + \frac{2}{3} = \frac{328}{231}$$

$\approx 1{,}42$

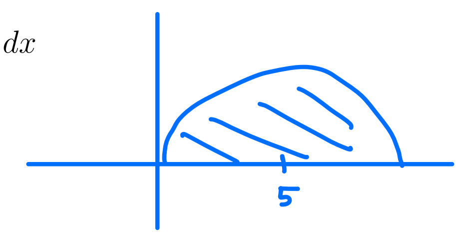
14. (10 points)

(a) Compute: $\int_0^{10} \sqrt{10x - x^2}\,dx$

$y = \sqrt{10x - x^2}$

$y^2 + x^2 - 10x + 25 = 25$

$(x-5)^2 + y^2 = 25$



$$= \frac{1}{2}\,25\pi = \frac{25\pi}{2}$$

(b) Compute: $\lim_{x\to 0} \frac{\int_0^x \arctan t\,dt}{x^2} = \text{"}\frac{0}{0}\text{"}$

$$\lim_{x\to 0} \frac{\text{ARCTAN}\,x}{2x} = \text{"}\frac{0}{0}\text{"} \quad \lim_{x\to 0} \frac{\frac{1}{1+x^2}}{2} = \frac{1}{2}$$

15. (10 points)

(a) Compute: $\displaystyle\int_1^2 \frac{x^2}{x^3+1}\,dx$

$u = x^3+1$

$du = 3x^2\,dx$

$$= \int \frac{\cancel{x^2}}{u}\frac{du}{3\cancel{x^2}} = \frac{1}{3}\ln|u| = \frac{1}{3}\ln|x^3+1|\Big|_1^2$$

$$= \frac{1}{3}(\ln 9 - \ln 2)$$

(b) Compute: $\displaystyle\int \frac{\sin x}{1+\cos^2 x}\,dx.$

$u = \cos x$

$du = -\sin x\,dx$

$$= \int \frac{\cancel{\sin x}}{1+u^2}\frac{du}{-\cancel{\sin x}} = -\text{ARCTAN } u$$

$$= -\text{ARCTAN}(\cos x) + C$$

16. (10 points)

(a) Find the area of the region bounded by the curves $y = 8 - x^2$ and $y = 2x$

$8 - x^2 = 2x$

$x^2 + 2x - 8 = 0$

$(x-2)(x+4) = 0$

$x = 2, -4$

$$\int_{-4}^{2} 8 - x^2 - 2x\,dx$$

$$= 8x - \frac{x^3}{3} - x^2\Big|_{-4}^{2} = 36$$

(b) Find the average value of $f(x) = \dfrac{x^2}{x^3+1}$ on $[1, 2]$

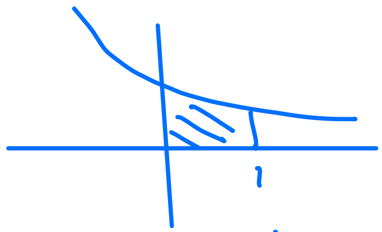
$$\text{AVE} = \frac{1}{b-a}\int_a^b \frac{x^2}{x^3+1}\,dx$$

$$= \frac{1}{2-1}\left(\frac{1}{3}(\ln(9) - \ln(2)\right)$$

17. (10 points) Find the volume of the solid generated by revolving the region bounded by

$$y = e^{-x}, \qquad y = 0, \qquad x = 0, \qquad x = 1$$

about the x-axis.



$$V = \int_0^1 \pi \left(e^{-x}\right)^2 dx$$

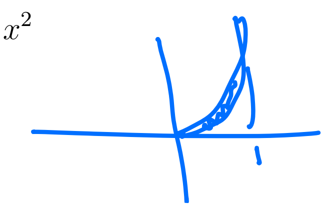
$$= \int_0^1 \pi e^{-2x} dx = \frac{\pi e^{-2x}}{-2} \Big|_0^1$$

$$= -\frac{\pi}{2}\left(e^{-2} - 1\right) = \frac{\pi}{2}\left(1 - e^{-2}\right)$$

18. (10 points) Find the volume of the solid generated by revolving the region bounded by

$$y = x^3 \qquad \text{and} \qquad y = x^2$$

about the y-axis.

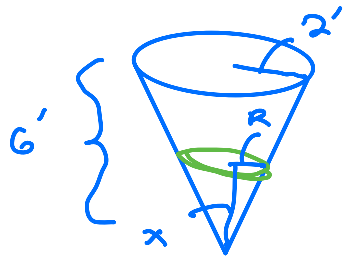


$$V = \int_0^1 2\pi x \left(x^2 - x^3\right) dx$$

$$= 2\pi \int_0^1 \left(x^3 - x^4\right) dx = 2\pi \left[\frac{x^4}{4} - \frac{x^5}{5}\right]_0^1$$

$$= 2\pi \left[\frac{1}{4} - \frac{1}{5}\right] = \frac{\pi}{10}$$

19. (10 points) A right circular conical tank of height 6 ft and radius of base 2 ft has its vertex at ground level and axis vertical. The tank is filled to 3 feet deep. of hot chocolate ($\rho = 64$ lb/ft^3). Find the work done in pumping all the hot chocolate out of the tank.



$W = F \cdot d$

$F = \text{WEIGHT} = V \cdot D = \pi R^2\, dx \cdot 64$

$d = 6 - x$

$\frac{R}{x} = \frac{2}{6}$

$R = \frac{x}{3}$

$W = \int_0^3 \pi \left(\frac{x^2}{9}\right)(6-x)(64)\, dx$

$= \frac{64\pi}{9} \int_0^3 6x^2 - x^3\, dx = \frac{64\pi}{9}\left[2x^3 - \frac{x^4}{4}\right]_0^3$

$= \frac{64\pi}{9}\left[54 - \frac{81}{4}\right] = 240\pi$ ft $\cdot$ Lb

20. (10 points) Indicate whether the following statements are true or false by circling the appropriate letter. A statement which is sometimes true and sometimes false should be marked false.

a) If $f(c) = L$, then $\lim_{x \to c} f(x) = L$. **T** **F**

b) $\sum_{j=1}^{25} 25 = 25$ **T** **F**

c) If $\int_a^b f(x)\, dx > 0$ then $f(x) > 0$ on $[a, b]$ **T** **F**

d) If f is continuous on $(a, b]$, then $f(x)$ has a maximum on $(a, b]$. **T** **F**

e) $1 + 1 = 2$ **T** **F**

FORMULA PAGE

$$\sin^2\theta + \cos^2\theta = 1$$

$$\tan^2\theta + 1 = \sec^2\theta$$

$$1 + \cot^2\theta = \csc^2\theta$$

$$\sin^2 x = \frac{1-\cos 2x}{2}$$

$$\cos^2 x = \frac{1+\cos 2x}{2}$$

$$\sin 2x = 2\sin x\cos x$$

$$(\sin x)' = \cos x$$

$$(\cos x)' = -\sin x$$

$$(\tan x)' = \sec^2 x$$

$$(\sec x)' = \sec x\tan x$$

$$(\csc x)' = -\csc x\cot x$$

$$(\cot x)' = -\csc^2 x$$

$$(e^x)' = e^x$$

$$(\ln x)' = \frac{1}{x}$$

$$(\arcsin x)' = \frac{1}{\sqrt{1-x^2}}$$

$$(\arctan x)' = \frac{1}{1+x^2}$$

$$(\operatorname{arcsec} x)' = \frac{1}{x\sqrt{x^2-1}}$$

$$\int \frac{du}{\sqrt{a^2-u^2}} = \arcsin\frac{u}{a} + C$$

$$\int \frac{du}{a^2+u^2} = \frac{1}{a}\arctan\frac{u}{a} + C$$

$$\int \frac{du}{u\sqrt{u^2-a^2}} = \frac{1}{a}\operatorname{arcsec}\frac{|u|}{a} + C$$

$$(\sinh x)' = \cosh x$$

$$(\cosh x)' = \sinh x$$

$$\sinh x = \frac{e^x - e^{-x}}{2}$$

$$\cosh x = \frac{e^x + e^{-x}}{2}$$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$f'(c) = \frac{f(b)-f(a)}{b-a}$$

$$f(c) = \frac{1}{b-a}\int_a^b f(x)dx$$

$$\sum_{j=1}^{n} j = \frac{n(n+1)}{2}$$

$$\sum_{j=1}^{n} j^2 = \frac{n(n+1)(2n+1)}{6}$$

$$\sum_{j=1}^{n} j^3 = \left[\frac{n(n+1)}{2}\right]^2$$

$$\Delta x = \frac{b-a}{n}$$

$$x_j = a + j\Delta x$$

$$\int_a^b f(x)dx = \lim_{n\to\infty}\sum_{j=1}^{n} f(x_j)\Delta x$$

$$f(x+\Delta x) \approx f(x) + f'(x)\Delta x$$

$$V = \int_a^b \pi\ \left(f(x)\right)^2\ dx$$

$$V = \int_a^b 2\pi\ x\ f(x)\ dx$$

$$W = \int_a^b F(x)\ dx$$

$$1 + 1 = 2$$