Math 223 Final

July 30, 2024

Name KEY

1	/10
2	/15
3	/15
4	/15
5	/10
6	/10
7	/10
8	/15
9	/15
10	/15
11	/15
12	/15
13	/10
14	/15
15	/15
Total	

Directions:

1. No books, notes, or dressing your dog up like a taco. You may use a calculator to do routine arithmetic computations. You may *not* use your calculator to store notes or formulas. You may not share a calculator with anyone.
2. You should show your work, and explain how you arrived at your answers. A correct answer with no work shown (except on problems which are completely trivial) will receive no credit. If you are not sure whether you have written enough, please ask.
3. You may not make more than one attempt at a problem. If you make several attempts, you must indicate which one you want counted, or you will be penalized.
4. You may leave as soon as you are finished, but once you leave the exam, you may not make any changes to your exam.

1. (10 points)

(a) Show that the following lines intersect:

$$x = 2 + 3t \qquad y = -4 - 2t \qquad z = -1 + 4t$$

$$x = 6 + 4t \qquad y = -2 + 2t \qquad z = -3 - 2t$$

$2 + 3t = 6 + 4s$

$-4 - 2t = -2 + 2s$

$8 + 4t = 4 - 4s$

$10 + 7t = 10$

$t = 0$

$s = -1$

$t = 0 \Rightarrow (2, -4, -1)$

$s = -1 \Rightarrow (2, -4, -1)$

$\boxed{\text{INTERSECT @ } (2, -4, -1)}$

(b) Find the equation of the plane containing the two lines in part (a).

$$\begin{array}{cc} & \begin{array}{ccc} i & j & k \end{array} \\ \vec{V}_1 = & \langle 3, -2, 4 \rangle \end{array} \quad 3 \ \ -2$$

$\vec{V}_2 = \langle 4, 2, -2 \rangle \quad 4 \ \ 2$

$\vec{N} = \langle -4, 22, 14 \rangle$

$-4x + 22y + 14z = -8 - 88 - 14$

$-4x + 22y + 14z = -110$

$\boxed{2x - 11y - 7z = 55}$

2. (15 points) For $\overrightarrow{r}(t) = \langle \frac{1}{2}t^2 + 1, \frac{8}{3}t^{3/2} + 1, 8t - 2\rangle$ for $0 \le t \le 1$. find

(a) $\vec{v}(t)$ at $t = 1$

$$\vec{r}\,'(t) = \langle t, 4t^{\frac{1}{2}}, 8\rangle\Big|_{t=1} = \boxed{\langle 1, 4, 8\rangle}$$

(b) $\vec{a}(t)$ at $t = 1$

$$\vec{r}\,''(t) = \langle 1, 2t^{-\frac{1}{2}}, 0\rangle\Big|_{t=1} = \boxed{\langle 1, 2, 0\rangle}$$

(c) $\vec{T}(t)$ at $t = 1$ (Hint: $\vec{T}(t) = \frac{\vec{v}(t)}{||\vec{v}(t)||}$)

$$\vec{T}(1) = \frac{\vec{v}(1)}{||\vec{v}(1)||} = \frac{\langle 1, 4, 8\rangle}{\sqrt{1+16+64}} = \boxed{\frac{1}{9}\langle 1, 4, 8\rangle}$$

(d) a_T at $t = 1$ (Hint: $a_T = \frac{\vec{v}\cdot\vec{a}}{||\vec{v}||}$)

$$\frac{\langle 1,4,8\rangle \cdot \langle 1,2,0\rangle}{9} = \frac{1+8+0}{9} = \boxed{1}$$

(e) Find the length of the curve for $0 \le t \le 1$.

$$s = \int_0^1 \sqrt{t^2 + 16t + 64}\, dt = \int_0^1 (t+8)\, dt$$

$$= \frac{t^2}{2} + 8t\Big|_0^1 = \frac{1}{2} + 8 = \boxed{\frac{17}{2}}$$

3. (15 points) Let $f(x, y) = x^4 + xy + y^2$.

(a) Find an equation of the plane tangent to the surface $z = f(x, y)$ at the point $(1, 2)$.

$f(1,2) = 1 + 2 + 4 = 7$

$f_x(x,y) = 4x^3 + y \Big|_{(1,2)} = 6$

$f_y(x,y) = x + 2y \Big|_{(1,2)} = 5$

$$\boxed{z - 7 = 6(x-1) + 5(y-2)}$$

(b) Find the directional derivative of $f(x, y)$ at the point $(1, 2)$ in the direction $\vec{a} = \langle 3, -4 \rangle$.

$D_a f = \nabla f \cdot \vec{u} = \langle 6,5 \rangle \cdot \dfrac{\langle 3,-4 \rangle}{5}$

$= \dfrac{18-20}{5} = \boxed{-\dfrac{2}{5}}$

4. (15 points) Find all local maxima, local minima, and saddle points for

$$f(x, y) = xy^2 - 6x^2 - 3y^2$$

$f_x = y^2 - 12x$

$f_y = 2xy - 6y$

$2xy - 6y = 0$

$2y(x-3) = 0$

$\underline{y = 0}$

$0 - 12x = 0$

$x = 0$

$\underline{x = 3}$

$y^2 - 36 = 0$

$y = \pm 6$

	f_{xx} -12	f_{yy} $2x-6$	f_{xy} $2y$	D	
$(0,0)$	-12	-6	0	72	MAX
$(3,6)$	-12	0	12	-144	SADDLE
$(3,-6)$	-12	0	-12	-144	SADDLE

5. (10 points) Find the maximum and minimum value of

$$f(x, y) = x^2 y$$

subject to the constraint $x^2 + y^2 = 3$

$\nabla f = \langle 2xy, x^2 \rangle$

$\nabla g = \langle 2x, 2y \rangle$

$2xy = \lambda(2x)$

$x^2 = \lambda 2y$

$2xy - \lambda(2x) = 0$

$2x(y - \lambda) = 0$

$\underline{x = 0}$

$y^2 = 3$

$y = \pm\sqrt{3}$

$\underline{y = \lambda}$

$x^2 = 2y^2$

$2y^2 + y^2 = 3$

$3y^2 = 3$

$y = \pm 1$

$x = \pm\sqrt{2}$

$f(0, \sqrt{3}) = 0$

$f(0, -\sqrt{3}) = 0$

$f(\sqrt{2}, 1) = 2$

$f(\sqrt{2}, -1) = -2$

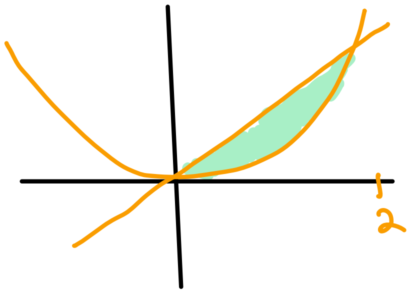
$f(-\sqrt{2}, 1) = 2$

$f(-\sqrt{2}, -1) = -2$

MAX OF 2

MIN OF -2

6. (10 points) Evaluate the integral $\iint_R (2y-x)\, dA$ where R is the region bounded by $y=x^2$ and $y=2x$



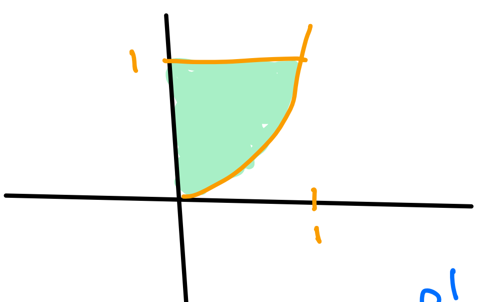
$$\int_0^2 \int_{x^2}^{2x} (2y-x)\, dy\, dx$$

$$= \int_0^2 y^2 - xy \Big|_{x^2}^{2x} \, dx$$

$$= \int_0^2 (4x^2 - 2x^2) - (x^4 - x^3)\, dx = \int_0^2 (2x^2 - x^4 + x^3)\, dx$$

$$= \frac{2x^3}{3} - \frac{x^5}{5} + \frac{x^4}{4}\Big|_0^2 = \frac{16}{3} - \frac{32}{5} + 4 = \boxed{\frac{44}{15}}$$

7. (10 points) $\int_0^1 \int_{x^4}^1 x^3 e^{y^2}\, dy\, dx$



$$\int_0^1 \int_0^{\sqrt[4]{y}} x^3 e^{y^2}\, dx\, dy$$

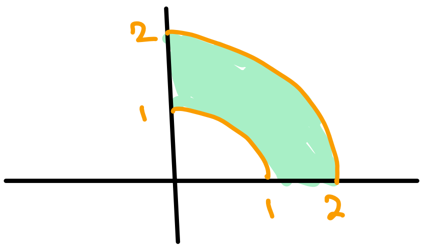
$$= \int_0^1 \frac{x^4}{4} e^{y^2} \Big|_0^{\sqrt[4]{y}} dy = \int_0^1 \frac{y}{4} e^{y^2}\, dy$$

$$= \frac{e^{y^2}}{8}\Big|_0^1 = \boxed{\frac{1}{8}(e-1)}$$

8. (15 points) Let R be the region in the first quadrant bounded by the circles: $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$

Evaluate the integral

$$\iint_R \sin(x^2 + y^2)\, dA$$



$$= \int_0^{\frac{\pi}{2}} \int_1^2 \sin(R^2)\, R\, dR\, d\theta$$

$$= \int_0^{\frac{\pi}{2}} -\frac{\cos(R^2)}{2} \Big|_1^2 = -\frac{1}{2} \int_0^{\frac{\pi}{2}} \cos(4) - \cos(1)\, d\theta$$

$$= \boxed{-\frac{\pi}{4}\left(\cos(4) - \cos(1)\right)}$$

9. (15 points) Let B be the region bounded by $0 \le x \le 1$, $0 \le y \le x$, and $0 \le z \le x + y$

Evaluate the integral

$$\iiint_B e^z \, dV$$

$$= \int_0^1 \int_0^x \int_0^{x+y} e^z \, dz \, dy \, dx$$

$$= \int_0^1 \int_0^x e^z \Big|_0^{x+y} dy \, dx$$

$$= \int_0^1 \int_0^x (e^x e^y - 1) \, dy \, dx$$

$$= \int_0^1 e^x e^y - y \Big|_0^x dx$$

$$= \int_0^1 (e^{2x} - x - e^x) \, dx$$

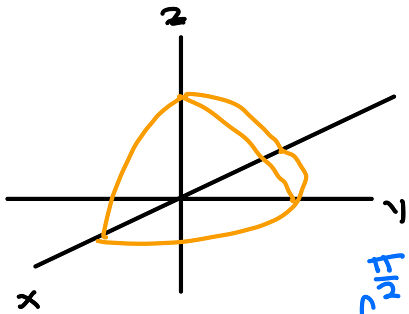
$$= \frac{e^{2x}}{2} - \frac{x^2}{2} - e^x \Big|_0^1$$

$$= \left(\frac{e^2}{2} - \frac{1}{2} - e\right) - \left(\frac{1}{2} - 1\right)$$

$$= \boxed{\frac{e^2}{2} - e}$$

10. (15 points) Evaluate the triple integral

$$\int_{-2}^{2} \int_{0}^{\sqrt{4-x^2}} \int_{0}^{\sqrt{4-x^2-y^2}} \sqrt{x^2+y^2+z^2}\; dz\; dy\; dx$$



$$= \int_0^{\frac{\pi}{2}} \int_0^{\pi} \int_0^{2} \rho\, \rho^2 \sin\varphi \; d\rho \; d\theta \; d\varphi$$

$$= \int_0^{\frac{\pi}{2}} \int_0^{\pi} \frac{\rho^4}{4}\Big|_0^2 \sin\varphi \; d\theta \; d\varphi$$

$$= \int_0^{\frac{\pi}{2}} 4\theta\Big|_0^{\pi} \sin\varphi \; d\varphi$$

$$= -4\pi \cos\varphi \Big|_0^{\frac{\pi}{2}} = -4\pi(0-1)$$

$$\boxed{= 4\pi}$$

11. (15 points) Compute

$$\iint_R xy(x^2+y^2)\, dA$$

where R is the region bounded by $x^2-y^2=-3, x^2-y^2=3, xy=1$, and $xy=4$.

$$u = x^2 - y^2 \qquad v = xy$$

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} 2x & -2y \\ y & x \end{vmatrix}$$

$$= 2x^2 + 2y^2 = 2(x^2+y^2)$$

$$\frac{\partial(x,y)}{\partial(u,v)} = \frac{1}{2(x^2+y^2)}$$

$$= \int_1^4 \int_{-3}^3 v \cancel{(x^2+y^2)} \frac{1}{2\cancel{(x^2+y^2)}}\, du\, dv$$

$$= \int_1^4 \left.\frac{uv}{2}\right|_{-3}^3 dv = \int_1^4 3v\, dv$$

$$= \left.\frac{3v^2}{2}\right|_1^4 = \frac{3(16-1)}{2} = \boxed{\frac{45}{2}}$$

12. (15 points)

(a) Let $G(x,y) = (x, y, xy)$, calculate $\vec{T_x}$, $\vec{T_y}$, $\vec{N}$ and $||\vec{N}||$.

$\vec{T}_x = \langle 1, 0, y \rangle$ 1 0

$\vec{T}_y = \langle 0, 1, x \rangle$ 0 1

$\vec{N} = \langle -y, -x, 1 \rangle$

$||\vec{N}|| = \sqrt{y^2 + x^2 + 1}$

(b) Calculate

$$\iint_S 1 \, dS$$

for where $S = G(D)$ where $D = \{(x,y) : x^2+y^2 \leq 1, x \geq 0, y \geq 0\}$

$$= \iint_D \sqrt{y^2 + x^2 + 1} \, dA$$

$$= \int_0^{\frac{\pi}{2}} \int_0^1 R\sqrt{R^2+1} \, dR \, d\theta$$

$$= \int_0^{\frac{\pi}{2}} \frac{\cancel{2}(R^2+1)^{\frac{3}{2}}}{3 \cdot \cancel{2}} \Big|_0^1 = \frac{1}{3}\int_0^{\frac{\pi}{2}} (2^{\frac{3}{2}} - 1) \, d\theta$$

$$= \boxed{\frac{\pi}{6}\left(2^{\frac{3}{2}} - 1\right)}$$

13. (10 points) Compute

$$\int_C 2x^2y\ dx + yz\ dy + (2z+y)\ dz$$

where C is the straight line from $(0,0,0)$ to $(1,2,3)$

$\vec{r}(t) = \langle t, 2t, 3t \rangle$

$= \int_0^1 \left(2(t^2)(2t) + (2t)(3t)(2) + (6t+2t)(3) \right) dt$

$= \int_0^1 (4t^3 + 12t^2 + 24t)\ dt$

$= t^4 + 4t^3 + 12t^2 \Big|_0^1 = \boxed{17}$

14. (15 points) Let $\vec{F}(x,y,z) = \langle yz + 2xy, xz + x^2, xy + 4z\rangle$.

(a) Show that $\vec{F}(x,y,z)$ is a conservative vector field, and find a potential function for $\vec{F}(x,y,z)$.

$\frac{\partial F_1}{\partial y} = z + 2x \qquad \frac{\partial F_2}{\partial z} = x \qquad \frac{\partial F_3}{\partial x} = y$

$\frac{\partial F_2}{\partial x} = z + 2x \qquad \frac{\partial F_3}{\partial y} = x \qquad \frac{\partial F_1}{\partial z} = y$

$\frac{\partial f}{\partial x} = F_1 = yz + 2xy \qquad f(x,y,z) = xyz + x^2y + g_1(y,z)$

$\frac{\partial f}{\partial y} = F_2 = xz + x^2 \qquad f(x,y,z) = xyz + x^2y + g_2(x,z)$

$\frac{\partial f}{\partial z} = F_3 = xy + 4z \qquad f(x,y,z) = xyz + 2z^2 + g_3(x,y)$

$f(x,y,z) = xyz + x^2y + 2z^2 + C$

(b) Evaluate

$$\int_C (yz + 2xy)\,dx + (xz + x^2))\,dy + (xy + 4z)\,dz$$

where C is the curve $\vec{r}(t) = \langle t,\ t^2,\ t^4 - 1\ \rangle$ for $0 \le t \le 1$.

$\vec{r}(0) = \langle 0,0,-1\rangle \qquad \vec{r}(1) = \langle 1,1,0\rangle$

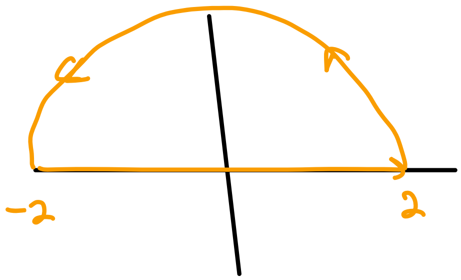
$= xyz + x^2y + 2z^2 \Big|_{(0,0,-1)}^{(1,1,0)}$

$= (0 + 1 + 0) - (0,0,2) = \boxed{-1}$

15. (15 points) Let R be the region bounded by the x-axis, and the semi-circle $y = \sqrt{4-x^2}$ and let C be the boundary curve of R oriented counterclockwise.

Evaluate the integral

$$\oint_C \overset{F_1}{(x^2y)}\,dx - \overset{F_2}{(y^2x)}\,dy$$



$$= \iint_R \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\,dA$$

$$= \iint_R (-y^2 - x^2)\,dA = -\iint_R (x^2+y^2)\,dA$$

$$= -\int_0^{\pi}\int_0^2 R^2\,R\,dR\,d\theta = -\int_0^{\pi} \frac{R^4}{4}\Big|_0^2\,d\theta$$

$$= -\int_0^{\pi} 4\,d\theta = -4\theta\Big|_0^{\pi} = \boxed{-4\pi}$$

FORMULA PAGE

$$\sin^2\theta + \cos^2\theta = 1$$

$$\tan^2\theta + 1 = \sec^2\theta$$

$$1 + \cot^2\theta = \csc^2\theta$$

$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$$

$$\sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$$

$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$

$$\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$$

$$\tan(\alpha+\beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}$$

$$\sin^2 x = \frac{1-\cos 2x}{2}$$

$$\cos^2 x = \frac{1+\cos 2x}{2}$$

$$\sin 2x = 2\sin x\cos x$$

$$\cos 2x = \cos^2 x - \sin^2 x$$

$$(\sin x)' = \cos x$$

$$(\cos x)' = -\sin x$$

$$(\tan x)' = \sec^2 x$$

$$(\sec x)' = \sec x\tan x$$

$$(\csc x)' = -\csc x\cot x$$

$$(\cot x)' = -\csc^2 x$$

$$(e^x)' = e^x$$

$$(\ln x)' = \frac{1}{x}$$

$$(\arcsin x)' = \frac{1}{\sqrt{1-x^2}}$$

$$(\arctan x)' = \frac{1}{1+x^2}$$

$$(\operatorname{arcsec} x)' = \frac{1}{|x|\sqrt{x^2-1}}$$

$$S = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\,dt$$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$f'(c) = \frac{f(b)-f(a)}{b-a}$$

$$f(c) = \frac{1}{b-a}\int_a^b f(x)dx$$

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$$

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$$

$$\sum_{k=1}^{n} k^3 = \left[\frac{n(n+1)}{2}\right]^2$$

$$\int \sec x\,dx = \ln|\sec x + \tan x| + C$$

$$\int \sec^3 x\,dx = \frac{1}{2}\left[\sec x\tan x + \ln|\sec x + \tan x|\right] + C$$

$$\int \csc x\,dx = \ln|\csc x - \cot x| + C$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = \sum_{n=0}^{\infty}\frac{x^n}{n!}, \text{ for all } x$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots = \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{(2n+1)!}, \text{ for all } x$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots = \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{(2n)!}, \text{ for all } x$$

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + \cdots = \sum_{n=0}^{\infty} x^n, -1 < x < 1$$

$$D = \frac{||\overrightarrow{PQ} \times \overrightarrow{v}||}{||\overrightarrow{v}||}$$

$$D = \frac{|ax_1 + by_1 + cz_1 - d|}{\sqrt{a^2+b^2+c^2}}$$

$$1 + 1 = 2$$