Math 223 Final

July 25, 2025

E.F.:

Name KEY

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Directions:

1. No books, notes, or affairs where there are kiss cams. You may use a calculator to do routine arithmetic computations. You may *not* use your calculator to store notes or formulas. You may not share a calculator with anyone.

2. You should show your work, and explain how you arrived at your answers. A correct answer with no work shown (except on problems which are completely trivial) will receive no credit. If you are not sure whether you have written enough, please ask.

3. You may not make more than one attempt at a problem. If you make several attempts, you must indicate which one you want counted, or you will be penalized.

4. You may leave as soon as you are finished, but once you leave the exam, you may not make any changes to your exam.

1. (15 points)

(a) Find the equation of the plane with normal vector $\overrightarrow{N} = \langle 2, 1, 6 \rangle$ and contains the point $(3, 8, 2)$

$2x + y + 6z = 6 + 8 + 12$

$2x + y + 6z = 26$

(b) Find the equation of the line that contains the $P(2, 2, 1)$ that is perpendicular to

$$2x + 3y - z = 5$$

$x = 2 + 2t$

$y = 2 + 3t$

$z = 1 - t$

$\vec{v} = \langle 2, 3, -1 \rangle$

$P(2, 2, 1)$

(c) Find the point of intersection of the line $x = 1$, $y = 1 + 2t$ and $z = 4t$ and the plane $x + y + z = 14$

$1 + 1 + 2t + 4t = 14$

$6t = 12 \quad t = 2$

$(1, 5, 8)$

2. (20 points) For the curve: $\vec{r}(t) = \langle 5t, 2\cos t, 2\sin t \rangle$.

(a) Find the arc length of the curve for $0 \le t \le \pi$

$$S = \int_0^{\pi} ||\vec{r}\,'(t)||\, dt = \int_0^{\pi} \sqrt{29}\, dt = \sqrt{29}\,\pi$$

$$\vec{r}\,'(t) = \langle 5, -2\sin t, 2\cos t \rangle \qquad ||\vec{r}\,'(t)|| = \sqrt{25 + 4\sin^2 t + 4\cos^2 t}$$
$$= \sqrt{29}$$

(b) Find the speed at $t = 1$

$$\frac{ds}{dt} = ||\vec{r}\,'(t)|| = \sqrt{29}$$

(c) $\vec{T}(t)$ at $t = \pi/2$. Hint: $\vec{T}(t) = \dfrac{\vec{r}\,'(t)}{||\vec{r}\,'(t)||}$

$$\vec{T}(t) = \frac{\langle 5, -2\sin t, 2\cos t \rangle}{\sqrt{29}}$$

$$= \frac{\langle 5, -2, 0 \rangle}{\sqrt{29}}$$

(d) $\kappa(t)$ at $t = \pi/2$. Hint: $\kappa(t) = \dfrac{||\vec{r}\,'(t) \times \vec{r}\,''(t)||}{||\vec{r}\,'(t)||^3}$

$$\vec{r}\,'(\tfrac{\pi}{2}) = \langle 5, -2, 0 \rangle \qquad \vec{r}\,''(t) = \langle 0, -2\cos t, -2\sin t \rangle$$

$$\vec{r}\,''(\tfrac{\pi}{2}) = \langle 0, 0, -2 \rangle \qquad \vec{r}\,'(\tfrac{\pi}{2}) \times \vec{r}\,''(\tfrac{\pi}{2}) = \langle 4, 10, 0 \rangle$$

$$||\vec{r}\,'(\tfrac{\pi}{2}) \times \vec{r}\,''(\tfrac{\pi}{2})|| = \sqrt{116}$$

$$K = \frac{\sqrt{116}}{(\sqrt{29})^3} = \frac{2\sqrt{29}}{(\sqrt{29})^3} = \frac{2}{29}$$

3. (10 points) Find each limit below or show that it does not exist.

(a) $\displaystyle\lim_{(x,y)\to(2,2)} \frac{x^2+xy-4x}{\sqrt{x+y}-2} \cdot \frac{\sqrt{x+y}+2}{\sqrt{x+y}+2}$

$$= \lim_{(x,y)\to(2,2)} \frac{x\cancel{[(x+y)-4]}(\sqrt{x+y}+2)}{\cancel{(x+y-4)}} =$$

$$2(2+2) = 8$$

(b) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{3x^4+5y^4}{x^4+3x^2y^2+y^4}$

$y=0$ $\quad \displaystyle\lim_{x\to 0} \frac{3x^4}{x^4} = 3$

$3 \neq 5$

$x=0$ $\quad \displaystyle\lim_{y\to 0} \frac{5y^4}{y^4} = 5$

D.N.E.

4. (15 points) For $f(x, y, z) = \sqrt{1 + x^2 + y^2 + z^2}$

(a) Find the gradient vector ∇f at $P(-1, -1, 1)$.

$$\nabla f = \left\langle \frac{x}{\sqrt{1+x^2+y^2+z^2}}, \frac{y}{\sqrt{1+x^2+y^2+z^2}}, \frac{z}{\sqrt{1+x^2+y^2+z^2}} \right\rangle \Big|_{(-1,-1,1)}$$

$$= \left\langle \frac{-1}{2}, \frac{-1}{2}, \frac{1}{2} \right\rangle$$

(b) Find the directional derivative of $f(x, y, z)$. at $P(-1, -1, 1)$ in the direction $\vec{v} = \langle 2, -2, -3 \rangle$.

$$D_{\vec{v}} f = \nabla f \cdot \vec{v} = \left\langle -\frac{1}{2}, -\frac{1}{2}, \frac{1}{2} \right\rangle \cdot \frac{\langle 2, -2, -3 \rangle}{\sqrt{17}}$$

$$= -\frac{3}{2\sqrt{17}}$$

(c) For $f(x, y, z)$ at $P(-1, -1, 1)$, in which direction is $f(x, y, z)$ increasing most rapidly?

$$\nabla f = \left\langle -\frac{1}{2}, -\frac{1}{2}, \frac{1}{2} \right\rangle$$

5. (15 points) Find all local maxima, local minima, and saddle points for

$$f(x,y) = x^2y - x^2 - 2y^2$$

$\frac{\partial f}{\partial x} = 2xy - 2x$

$\frac{\partial f}{\partial y} = x^2 - 4y$

$2x(y-1) = 0$

$x = 0 \quad y = 1$

$\underline{x = 0}$

$-4y = 0$

$y = 0$

$y = 1$

$x^2 = 4$

$x = \pm 2$

(x,y)	f_{xx} $2y-2$	f_{yy} -4	f_{xy} $2x$	D	
$(0,0)$	-2	-4	0	8	MAX
$(2,1)$	0	-4	4	-16	SADDLE
$(-2,1)$	0	-4	-4	-16	SADDLE

6. (15 points) Find the maximum and minimum value of

$$f(x,y) = x^2 + 6y$$

subject to the constraint $x^2 + y^2 = 25$

$\nabla f = \langle 2x, 6 \rangle$ $\nabla g = \langle 2x, 2y \rangle$

$2x = \lambda(2x)$ $6 = \lambda 2y$

$2x(1-\lambda) = 0$

$x = 0$ $\lambda = 1$

$\underline{x = 0}$

$y = \pm 5$

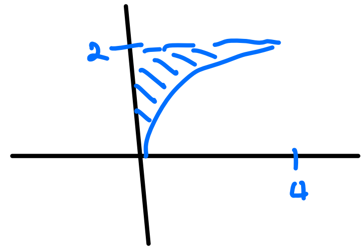
$\underline{\lambda = 1}$

$y = 3$

$x = \pm 4$

(x,y)	$f(x,y)$	
$(0,5)$	30	
$(0,-5)$	-30	MIN
$(4,3)$	34	MAX
$(-4,3)$	34	MAX

7. (10 points) $\displaystyle\int_0^4 \int_{\sqrt{x}}^2 \frac{1}{y^3+1}\, dy\, dx$



$\sqrt{x} \le y \le 2$

$0 \le x \le 4$ $\implies$ $0 \le x \le y^2$

$0 \le y \le 2$

$$\int_0^2 \int_0^{y^2} \frac{1}{y^3+1}\, dx\, dy = \int_0^2 \frac{y^2}{y^3+1}\, dy$$

$$= \left. \frac{\ln|y^3+1|}{3} \right|_0^2 = \frac{\ln 9}{3}$$

8. (10 points) $\displaystyle\int_{-2}^{2}\int_{0}^{\sqrt{4-y^2}} \cos(x^2+y^2)\,dx\,dy$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{0}^{2} R\cos(R^2)\,dR\,d\theta$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left.\frac{\sin(R^2)}{2}\right|_{0}^{2} d\theta = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\sin 4}{2}\,d\theta$$

$$= \frac{\pi \sin(4)}{2}$$

9. (10 points) Evaluate

$$\int_{-2}^{2}\int_{0}^{y}\int_{0}^{x^2}(6y+2z)\ dz\ dx\ dy$$

$$= \int_{-2}^{2}\int_{0}^{y} (6yz+z^2)\Big|_{0}^{x^2}\ dx\ dy$$

$$= \int_{-2}^{2}\int_{0}^{y} (6yx^2+x^4)\ dx\ dy$$

$$= \int_{-2}^{2} 2yx^3+\frac{x^5}{5}\Big|_{0}^{y}\ dy$$

$$= \int_{-2}^{2} 2y^4+\frac{y^5}{5}\ dy = \frac{2y^5}{5}-\frac{y^6}{30}\Big|_{-2}^{2}$$

$$= \left(\frac{64}{5}-\frac{64}{30}\right)-\left(-\frac{64}{5}-\frac{64}{30}\right)$$

$$= \frac{128}{5}$$

10. (10 points) Evaluate

$$\int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{x+y} \sqrt{x^2+y^2}\; dz\; dy\; dx$$

$$= \int_0^{\frac{\pi}{2}} \int_0^1 \int_0^{r(\cos\theta+\sin\theta)} R \quad R \; dz \; dR \; d\theta$$

$$= \int_0^{\frac{\pi}{2}} \int_0^1 R^3 (\cos\theta + \sin\theta) \; dR \; d\theta$$

$$= \int_0^{\frac{\pi}{2}} \frac{R^4}{4}\Big|_0^1 (\cos\theta + \sin\theta) \quad d\theta$$

$$= \frac{1}{4}\Big[\sin\theta - \cos\theta \Big]_0^{\frac{\pi}{2}} = \frac{1}{4}\Big[1 + 1 \Big] = \frac{1}{2}$$

11. (10 points) Evaluate

$$\iiint_B \frac{1}{\sqrt{x^2+y^2}}\, dV$$

where B is the region between the spheres

$$x^2+y^2+z^2=1 \qquad x^2+y^2+z^2=4$$

with $x \geq 0$, $y \geq 0$ and $z \geq 0$.

$$= \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \int_1^2 \frac{1}{\sqrt{\rho^2\cos^2\theta\sin^2\varphi + \rho^2\sin^2\theta\sin^2\varphi}}\, \rho^2 \sin\varphi \, d\rho \, d\varphi \, d\theta$$

$$= \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \int_1^2 \frac{\rho^{\cancel{2}} \cancel{\sin\varphi}}{\cancel{\rho} \cancel{\sin\varphi}}\, d\rho \, d\varphi \, d\theta$$

$$= \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \frac{\rho^2}{2}\Big|_1^2 \, d\varphi \, d\theta$$

$$= \frac{3}{2}\left(\frac{\pi}{2}\right)\left(\frac{\pi}{2}\right) = \frac{3\pi^2}{8}$$

12. (15 points)

(a) Find the Jacobian $\dfrac{\partial(x,y)}{\partial(u,v)}$ for $u = x^2 - y^2$and $v = xy$

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} 2x & -2y \\ y & x \end{vmatrix} = 2x^2+2y^2$$

$$\frac{\partial(x,y)}{\partial(u,v)} = \frac{1}{2(x^2+y^2)}$$

(b) Compute the value of

$$\iint_R xy(x^2+y^2)\, dA$$

where R is the region bounded by

$$-3 \le x^2 - y^2 \le 3 \qquad 1 \le xy \le 4$$

$$\int_{-3}^{3}\int_{1}^{4} v\,\cancel{(x^2+y^2)}\,\frac{1}{2\cancel{(x^2+y^2)}}\,dv\,du$$

$$= \int_{-3}^{3} \frac{v^2}{4}\Big|_1^4\, du = \int_{-3}^{3} \frac{15}{4}\, du =$$

$$\frac{15}{4}(3+3) = \frac{90}{4} = \frac{45}{2}$$

13. (10 points) Compute:

$$\int_C 3x \; ds$$

where C is the curve $y = x^2$ from $(0,0)$ to $(3,9)$.

$\vec{r}(t) = \langle t, t^2 \rangle \qquad \vec{r}\,'(t) = \langle 1, 2t \rangle$

$\|\vec{r}\,'(t)\| = \sqrt{1+4t^2}$

$$= \int_0^3 3t\sqrt{1+4t^2}\; dt$$

$$\left.\frac{(1+4t^2)^{\frac{3}{2}}}{4}\right|_0^3 = \frac{1}{4}\left(37^{\frac{3}{2}} - 1\right)$$

$F_1 \qquad F_2$

14. (15 points) Let $\vec{F}(x,y) = \langle 10x^4 + 2xy^3, 3x^2y^2 \rangle$.

(a) Show that $\vec{F}(x,y)$ is a conservative vector field.

$$\frac{\partial F_2}{\partial x} = 6xy^2 \qquad \frac{\partial F_1}{\partial y} = 6xy^2$$

$$= \quad \text{CONSERVATIVE!}$$

(b) Find a potential function for $\vec{F}(x,y)$.

$$f(x,y) = \int (10x^4 + 2xy^3)\,dx = 2x^5 + x^2y^3 + g_1(y)$$

$$f(x,y) = \int 3x^2y^2\,dy = x^2y^3 + g_2(x)$$

$$f(x,y) = 2x^5 + x^2y^3 + C$$

(c) Evaluate

$$\int_C (10x^4 + 2xy^3)\,dx + (3x^2y^2)\,dy,$$

where C is the part of the curve $x^5 - 5x^2y^2 - 7x^2 = 0$ from $(3,-2)$ to $(3,2)$.

$$= 2x^5 + x^2y^3 \Big|_{(3,-2)}^{(3,2)}$$

$$= 2(3)^5 + 72 - (2(3)^5 - 72)$$

$$= 144$$

15. (15 points) Find the area of the part of the surface $z = 2xy - 1$, that lies in the cylinder $x^2 + y^2 = 9$.

$$\vec{N} = \langle f_x, f_y, 1 \rangle = \langle 2y, 2x, 1 \rangle$$

$$\|\vec{N}\| = \sqrt{4x^2 + 4y^2 + 1}$$

$$SA = \iint_S dS = \iint_D \sqrt{4x^2+4y^2+1}\, dA$$

$$= \int_0^{2\pi} \int_0^3 R\sqrt{4R^2+1}\, dR\, d\theta$$

$$\int_0^{2\pi} \frac{(4R^2+1)^{\frac{3}{2}}}{12} \Big|_0^3 \, d\theta$$

$$\frac{\pi}{6}\left[37^{\frac{3}{2}} - 1\right]$$

16. (15 points) Let R be the square with vertices $(1,1), (-1,1), (-1,-1)$ and $(1,-1)$ and C be the boundary curve of R oriented counterclockwise.

Evaluate the integral

$$\oint_C \overset{F_1}{(x + y + \arcsin x)}\, dx + \overset{F_2}{(y - x + e^{y^2})}\, dy$$

$$= \iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) dA = \iint_R (-1 - 1)\, dA$$

$$= -2 \iint_R dA = -2\,(\text{AREA}) = (-2)(2)(2)$$

$$= -8$$

FORMULA PAGE

$$\sin^2\theta + \cos^2\theta = 1$$

$$\tan^2\theta + 1 = \sec^2\theta$$

$$1 + \cot^2\theta = \csc^2\theta$$

$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$$

$$\sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$$

$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$

$$\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$$

$$\tan(\alpha+\beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}$$

$$\sin^2 x = \frac{1-\cos 2x}{2}$$

$$\cos^2 x = \frac{1+\cos 2x}{2}$$

$$\sin 2x = 2\sin x\cos x$$

$$(\sin x)' = \cos x$$

$$(\cos x)' = -\sin x$$

$$(\tan x)' = \sec^2 x$$

$$(\sec x)' = \sec x\tan x$$

$$(\csc x)' = -\csc x\cot x$$

$$(\cot x)' = -\csc^2 x$$

$$(e^x)' = e^x$$

$$(\ln x)' = \frac{1}{x}$$

$$(\arcsin x)' = \frac{1}{\sqrt{1-x^2}}$$

$$(\arctan x)' = \frac{1}{1+x^2}$$

$$(\operatorname{arcsec} x)' = \frac{1}{|x|\sqrt{x^2-1}}$$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$f'(c) = \frac{f(b)-f(a)}{b-a}$$

$$f(c) = \frac{1}{b-a}\int_a^b f(x)dx$$

$$\int \sec x\,dx = \ln|\sec x + \tan x| + C$$

$$\int \sec^3 x\,dx = \frac{1}{2}\left[\sec x\tan x + \ln|\sec x + \tan x|\right] + C$$

$$\int \csc x\,dx = \ln|\csc x - \cot x| + C$$

$$1 + 1 = 2$$