Math 223 Final
July 24, 2014

Name__________________________________________

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Directions:

1. No books, notes, or evil looks. You may use a calculator to do routine arithmetic computations. You may not use your calculator to store notes or formulas. You may not share a calculator with anyone.

2. You should show your work, and explain how you arrived at your answers. A correct answer with no work shown (except on problems which are completely trivial) will receive no credit. If you are not sure whether you have written enough, please ask.

3. You may not make more than one attempt at a problem. If you make several attempts, you must indicate which one you want counted, or you will be penalized.

4. You may leave as soon as you are finished, but once you leave the exam, you may not make any changes to your exam.
1. (10 points)

(a) Show that the following lines intersect:

\[ x = 2 + 3t \quad y = -4 - 2t \quad z = -1 + 4t \]

\[ x = 6 + 4t \quad y = -2 + 2t \quad z = -3 - 2t \]

(b) Find the equation of the plane containing the two lines in part (a).
2. (20 points) For $\mathbf{r}(t) = \langle e^t \cos t, e^t \sin t, e^t \rangle$, find

(a) $\mathbf{v}(t)$ at $t = 0$

(b) $\mathbf{a}(t)$ at $t = 0$

(c) $\mathbf{T}(t)$ at $t = 0$ (Hint: $\mathbf{T}(t) = \frac{\mathbf{v}(t)}{|\mathbf{v}(t)|}$)

continued on next page.
(d) $\vec{N}(t)$ at $t = 0$ (Hint: $\vec{N}(t) = \frac{\vec{T}'(t)}{|\vec{T}'(t)|}$)

(e) $a_T$ at $t = 0$ (Hint: $a_T = \frac{\vec{v} \cdot \vec{a}}{|\vec{v}|}$)

(f) $a_N$ at $t = 0$ (Hint: $a_N = \frac{|\vec{v} \times \vec{a}|}{|\vec{v}|}$)

(g) Find the curvature $\kappa$ at $t = 0$. (Hint: $\kappa = \frac{|\vec{v} \times \vec{a}|}{|\vec{v}|^3}$)
3. (15 points) Let $f(x, y) = x^4 + xy + y^2$.

(a) Find an equation of the plane tangent to the surface $z = f(x, y)$ at the point $(1, 2)$.

(b) Find the directional derivative of $f(x, y)$ at the point $(1, 2)$ in the direction $\vec{a} = \langle 3, -4 \rangle$. 
4. (15 points) Find the minimum value of

\[ f(x, y, z) = xy + 2z \]

subject to the constraint \( x^2 + y^2 + z^2 = 36 \)
5. (15 points) Find all local maxima, local minima, and saddle points for

\[ f(x, y) = 3x - x^3 - 3xy^2 \]
6. (15 points)

(a) Evaluate the integral \( \int \int _R y \, dA \) where \( R \) is the region in the first quadrant enclosed by \( x^2 + y^2 = 25 \) and \( x + y = 5 \)

(b) \( \int _0^4 \int _{\sqrt{y}}^2 e^{x^3} \, dx \, dy \)
7. (15 points) Let \( R \) be the region in the first quadrant bounded by the circles: \( x^2 + y^2 = 1, x^2 + y^2 = 4 \), the line \( y = x \) and the \( x \)-axis.

Evaluate the integral

\[
\iint_{R} \frac{1}{\sqrt{4 - x^2 - y^2}} \, dA
\]
8. (10 points) Set-up (but do not evaluate) the triple integral \( \iiint_T xyz^2 \, dV \)
where \( T \) is the region bounded by \( x = 0, y = 0, z = 0 \) and the plane \( 2x + y + 3z = 6 \)

\[
\begin{align*}
\text{(a)} \quad \iiint_T xyz^2 \, dV &= \iiint_D \, dz \, dy \, dx \\
\text{(b)} \quad \iiint_T xyz^2 \, dV &= \iiint_D \, dx \, dy \, dz.
\end{align*}
\]
9. (15 points) Evaluate the triple integral

\[ \int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int_{-\sqrt{4-x^2-y^2}}^{\sqrt{4-x^2-y^2}} \sqrt{x^2 + y^2 + z^2} \, dz \, dy \, dx \]
10. (15 points)

(a) Let \( G(u, v) = (2u + 1, u - v, 3u + v) \), calculate \( \vec{T}_u, \vec{T}_v, \vec{n} \) and \( ||\vec{n}|| \).

(b) Calculate

\[
\int \int_S yz \, dS
\]

for where \( S = G(D) \) where \( D = \{(u, v) : 0 \leq u \leq 2, 0 \leq v \leq 1\} \).
11. (15 points) Compute

$$\int\int_R (x + y)^2 \sin^2(x - y) \, dA$$

where $R$ is the region bounded by $x + y = 1, x + y = 3, x - y = 1$ and $x - y = -1$
12. (10 points) Compute

\[ \int_C 2x^2 y \, dx + yz \, dy + (3z + y) \, dz \]

where \( C \) is the straight line from \((0, 0, 0)\) to \((1, 2, 3)\)
13. (15 points) Let \( \vec{F}(x, y) = (2xy^2, 2x^2y + 3y^2) \).

(a) Show that \( \vec{F}(x, y) \) is a conservative vector field, and find a potential function for \( \vec{F}(x, y) \).

(b) Evaluate
\[
\int_C 2xy^2 \, dx + (2x^2y + 3y^2) \, dy
\]
where \( C \) is the curve from \((1, 1)\) to \((0, \sqrt{2})\) along the circle \( x^2 + y^2 = 2 \).
14. (15 points) Let $R$ be the region in the first quadrant bounded by the $y$-axis, the line $y = x$ and the parabola $y = 2 - x^2$, and let $C$ be the boundary curve of $R$ oriented counterclockwise. Evaluate the integral

$$\oint_C (y - xy + \tan^3 x) \, dx + (x + \ln y) \, dy$$
FORMULA PAGE

\[ \sin^2 \theta + \cos^2 \theta = 1 \]
\[ \tan^2 \theta + 1 = \sec^2 \theta \]
\[ 1 + \cot^2 \theta = \csc^2 \theta \]
\[ \sin(\alpha + \beta) = \sin \alpha \cos \beta + \cos \alpha \sin \beta \]
\[ \sin(\alpha - \beta) = \sin \alpha \cos \beta - \cos \alpha \sin \beta \]
\[ \cos(\alpha + \beta) = \cos \alpha \cos \beta - \sin \alpha \sin \beta \]
\[ \cos(\alpha - \beta) = \cos \alpha \cos \beta + \sin \alpha \sin \beta \]
\[ \tan(\alpha + \beta) = \frac{\tan \alpha + \tan \beta}{1 - \tan \alpha \tan \beta} \]
\[ \sin^2 x = \frac{1 - \cos 2x}{2} \]
\[ \cos^2 x = \frac{1 + \cos 2x}{2} \]
\[ \sin 2x = 2 \sin x \cos x \]
\[ (\sin x)' = \cos x \]
\[ (\cos x)' = -\sin x \]
\[ (\tan x)' = \sec^2 x \]
\[ (\sec x)' = \sec x \tan x \]
\[ (\csc x)' = -\csc x \cot x \]
\[ (\cot x)' = -\csc^2 x \]
\[ (e^x)' = e^x \]
\[ (\ln x)' = \frac{1}{x} \]
\[ (\arcsin x)' = \frac{1}{\sqrt{1 - x^2}} \]
\[ (\arctan x)' = \frac{1}{1 + x^2} \]
\[ (\text{arcsec } x)' = \frac{1}{|x|\sqrt{x^2 - 1}} \]
\[ x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \]
\[ f'(c) = \frac{f(b) - f(a)}{b - a} \]