Math 122 Test 2

SI: ☐ EF: ☐

October 14, 2025

Name KEY

1	
2	
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4	
5	
Total	

Directions:

1. No books, notes or music parts with only one note. You may use a calculator to do routine arithmetic computations. You may *not* use your calculator to store notes or formulas. You may not share a calculator with anyone.

2. You should show your work, and explain how you arrived at your answers. A correct answer with no work shown (except on problems which are completely trivial) will receive no credit. If you are not sure whether you have written enough, please ask.

3. You may not make more than one attempt at a problem. If you make several attempts, you must indicate which one you want counted, or you will be penalized.

4. You may leave as soon as you are finished, but once you leave the exam, you may not make any changes to your exam.

1. (20 points)

   (a) Let $p(x) = ax + b$ for $0 \leq x \leq 5$ be a probability density function and
   have mean $\mu = \frac{35}{12}$

   Find the values of a and b.

$$\int_0^5 ax+b \, dx = a\frac{x^2}{2} + bx\Big|_0^5 = \frac{a(25)}{2} + 5b = 1$$

$$25a + 10b = 2$$

$$\int_0^5 x(ax+b)\, dx = \frac{ax^3}{3} + \frac{bx^2}{2}\Big|_0^5 = \frac{a(125)}{3} + \frac{b(25)}{2} = \frac{35}{12}$$

$$100a + 30b = 7$$
$$100a + 40b = 8$$
$$-10b = -1$$

$$\boxed{b = \frac{1}{10} \qquad a = \frac{1}{25}}$$

   (b) Find the surface area of the solid of revolution if $y = \sqrt{7-x}$ for $0 \leq x \leq 3$ is rotated about the x-axis.

$$SA = \int_a^b 2\pi f(x)\sqrt{1+(f'(x))^2}\, dx$$

$$f(x) = \sqrt{7-x} \qquad f'(x) = \frac{1}{2}(7-x)^{-\frac{1}{2}}(-1) \qquad (f'(x))^2 = \frac{1}{4(7-x)}$$

$$1+(f'(x))^2 = 1 + \frac{1}{4(7-x)} = \frac{4(7-x)+1}{4(7-x)} = \frac{29-4x}{4(7-x)}$$

$$SA = \int_0^3 \cancel{2}\pi \cancel{\sqrt{7-x}} \sqrt{\frac{29-4x}{\cancel{4}(\cancel{7-x})}}\, dx = -\frac{2\pi}{12}(29-4x)^{\frac{3}{2}}\Big|_0^3$$

$$= -\frac{\pi}{6}\left[17^{\frac{3}{2}} - 29^{\frac{3}{2}}\right] = \boxed{\frac{\pi}{6}\left[29^{\frac{3}{2}} - 17^{\frac{3}{2}}\right]}$$

2. (20 points)

(a) Find the centroid of the region lying between $y = x^2$ and $y = 0$ for $0 \leq x \leq 2$.

$$m = \int_0^2 x^2\, dx = \frac{x^3}{3}\Big|_0^2 = \frac{8}{3}$$

$$\boxed{\left(\frac{3}{2}, \frac{6}{5}\right)}$$

$$m_y = \int_0^2 x(x^2)\, dx = \frac{x^4}{4}\Big|_0^2 = 4$$

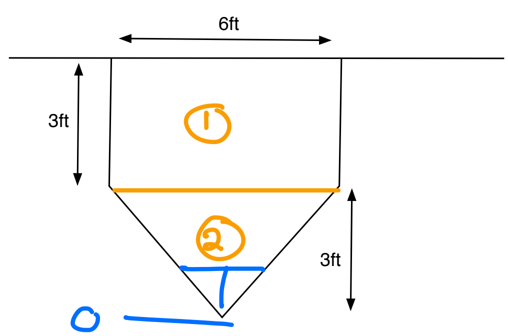
$$m_x = \int_0^2 \frac{x^4}{2}\, dx = \frac{x^5}{10}\Big|_0^2 = \frac{32}{10} = \frac{16}{5}$$

$$\bar{x} = \frac{m_y}{m} = \frac{4}{\frac{8}{3}} = \frac{12}{8} = \frac{3}{2} \qquad m_y = \frac{m_x}{m} = \frac{\cancel{16}^{\,2}}{5}\cdot\frac{3}{\cancel{8}} = \frac{6}{5}$$

(b) Find the force on the vertical flat plate shown below that is submerged in water (ρ =62.4 lb/ ft^3). The top of the plate is at the surface of the water.

$$\frac{x}{w} = \frac{3}{6}$$

$$w = 2x$$



$$F_1 = \int_0^3 (62.4)(x)(6)\, dx = 62.4\,(3x^2)\Big|_0^3 = (62.4)(27)$$

$$= 1{,}684.8 \text{ Lbs}$$

$$F_2 = \int_0^3 (62.4)(6-x)(2x)\, dx = 62.4\int_0^3 12x - 2x^2\, dx$$

$$= 62.4\left[6x^2 - \frac{2x^3}{3}\right]_0^3 = 62.4\left[54-18\right] = 2{,}246.4 \text{ Lbs}$$

$$F = 1{,}684.8 + 2{,}246.4 = \boxed{3931.2 \text{ Lbs}}$$

3. (20 points)

(a) Solve $(x+2)y' = x^2y - 4y; \qquad y(0) = 3$

$$\int \frac{1}{y}\, dy = \int \frac{x^2-4}{x+2}\, dx = \int x-2\, dx$$

$$\ln y = \frac{x^2}{2} - 2x + c$$

$$Y = c\, e^{\frac{x^2}{2} - 2x} \qquad 3 = c\, e^0 \qquad c = 3$$

$$\boxed{Y = 3\, e^{\frac{x^2}{2} - 2x}}$$

(b) Use Euler's method with $h = .5$ to approximate $y(2)$ if

$$x\frac{dy}{dx} = y^2 + 1 \qquad y(1) = 0$$

$$\frac{dy}{dx} = \frac{y^2+1}{x}$$

x	y	F(x,y)	hF(x,y)	y + h F(x,y)
1	0	1	.5	.5
1.5	.5	.8333	.4166	.9166
2	.9166			

$Y(2) \approx .9166$

4. (20 points)

(a) Konrad takes his 72^o F bagel outside where the temperature is -20^oF. At 12:00 PM, the bagel's temperature has dropped to 60°F, and by 12:02 PM, it has cooled further to 50°F. At what time did Konrad originally take the bagel outside?

$$\frac{dy}{dt} = -k(y - T_0) \qquad y = T_0 + ce^{-kt}$$

$$T_0 = -20 \qquad y(0) = 60 \qquad y(2) = 50$$

$$60 = -20 + c\cancel{e^0} \qquad c = 80 \qquad 50 = -20 + 80e^{-k(2)}$$

$$k = 0.0668$$

$$72 = -20 + 80e^{-.0668t} \qquad t = -2.09 \text{ min.}$$

$$\boxed{\approx 11{:}58 \text{ AM}}$$

(b) Typhoid Jonathan has sparked a flu outbreak at CWRU. After one week, 200 students are infected. By the end of the second week, the number of infected students has increased to 500.

$$\frac{dy}{dt} = ky(1 - \frac{y}{A})$$

with carrying capacity of 1000. How many people did Jonathan originally infected?

$$A = 1000$$

$$k =$$

$$y_0 = 200$$

$$B = \frac{y_0}{y_0 - A} = \frac{200}{200-1000} = -\frac{1}{4}$$

$$y = \frac{1000}{1+4e^{-kt}}$$

$$y(1) = \frac{1000}{1+4e^{-k}} = 500$$

$$k = 1.386$$

$$y(-1) = \frac{1000}{1+4e^{1.386}} \approx \boxed{59 \text{ STUDENTS}}$$

5. (20 points)

(a) Solve $x^2y' + 3xy = e^{-x^2}$ $\qquad y(1) = 0$

$$y' + \frac{3}{x}y = \frac{e^{-x^2}}{x^2} \qquad P(x) = \frac{3}{x} \qquad Q(x) = \frac{e^{-x^2}}{x^2}$$

$$\alpha(x) = e^{\int P(x)\,dx} = e^{\int \frac{3}{x}\,dx} = e^{3\ln x} = x^3$$

$$y = \frac{1}{\alpha(x)}\left[\int \alpha(x) Q(x)\,dx + c\right] = \frac{1}{x^3}\left[\int x^3 \frac{e^{-x^2}}{x^2}\,dx + c\right]$$

$$= x^{-3}\left[\int x e^{-x^2}\,dx + c\right] = x^{-3}\left[-\frac{e^{-x^2}}{2} + c\right]$$

$$0 = -\frac{e^{-1}}{2} + c \qquad c = \frac{e^{-1}}{2} \qquad \boxed{y = x^3\left[-\frac{e^{-x^2}}{2} + \frac{e^{-1}}{2}\right]}$$

(b) Vincent is brewing one of his sinister concoctions. He starts with a tank containing 20 gallons of water and 5 ounces of eye-of-newt. A solution with a concentration of 3 ounces of eye-of-newt per gallon flows into the tank at a rate of 4 gallons per minute. At the same time, the well-stirred mixture is draining from the tank at the same rate of 4 gallons per minute. How much eye-of-newt is in the tank after 10 minutes?

$$\frac{dy}{dt} = \text{RATE}_{IN} - \text{RATE}_{OUT}$$

$$y(0) = 5 \qquad \frac{dy}{dt} = \frac{4\,\text{gal}}{\text{m.}}\frac{3\,\text{oz}}{\text{gal}} - \frac{4\,\text{gal}}{\text{min}}\frac{y}{20}$$

$$\frac{dy}{dt} = 12 - \frac{1}{5}y \qquad \frac{dy}{dt} + \frac{1}{5}y = 12 \qquad P(t) = \frac{1}{5} \qquad Q(t) = 12$$

$$\alpha(t) = e^{\int \frac{1}{5}dt} = e^{\frac{1}{5}t}$$

$$y = \frac{1}{e^{\frac{1}{5}t}}\left[\int e^{\frac{1}{5}t}\cdot 12\,dt + c\right] = e^{-\frac{1}{5}t}\left[60e^{\frac{1}{5}t} + c\right]$$

$$y = 60 + ce^{-\frac{1}{5}t} \qquad 5 = 60 + ce^{0} \qquad c = -55$$

$$y = 60 - 55e^{-\frac{1}{5}t} \qquad y(10) = 60 - 55e^{-\frac{1}{5}(10)} = \boxed{52.5\ \text{oz.}}$$

FORMULA PAGE

$$\sin^2\theta + \cos^2\theta = 1$$

$$\tan^2\theta + 1 = \sec^2\theta$$

$$1 + \cot^2\theta = \csc^2\theta$$

$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$$

$$\sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$$

$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$

$$\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$$

$$\tan(\alpha+\beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}$$

$$\sin^2 x = \frac{1-\cos 2x}{2}$$

$$\cos^2 x = \frac{1+\cos 2x}{2}$$

$$\sin 2x = 2\sin x\cos x$$

$$(\sin x)' = \cos x$$

$$(\cos x)' = -\sin x$$

$$(\tan x)' = \sec^2 x$$

$$(\sec x)' = \sec x\tan x$$

$$(\csc x)' = -\csc x\cot x$$

$$(\cot x)' = -\csc^2 x$$

$$(e^x)' = e^x$$

$$(\ln x)' = \frac{1}{x}$$

$$(\arcsin x)' = \frac{1}{\sqrt{1-x^2}}$$

$$(\arctan x)' = \frac{1}{1+x^2}$$

$$(\operatorname{arcsec} x)' = \frac{1}{|x|\sqrt{x^2-1}}$$

$$\sinh x = \frac{e^x - e^{-x}}{2}$$

$$\cosh x = \frac{e^x + e^{-x}}{2}$$

$$\cosh^2 x - \sinh^2 x = 1$$

$$(\sinh x)' = \cosh x$$

$$(\cosh x)' = \sinh x$$

$$(\text{arcsinh } x)' = \frac{1}{\sqrt{x^2+1}}$$

$$(\text{arccosh } x)' = \frac{1}{\sqrt{x^2-1}}$$

$$(\text{arctanh } x)' = \frac{1}{1-x^2}$$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$f'(c) = \frac{f(b)-f(a)}{b-a}$$

$$f(c) = \frac{1}{b-a}\int_a^b f(x)dx$$

$$\int \sec x\,dx = \ln|\sec x + \tan x| + C$$

$$\int \sec^3 x\,dx = \frac{1}{2}\left[\sec x\tan x + \ln|\sec x + \tan x|\right] + C$$

$$\int \csc x\,dx = \ln|\csc x - \cot x| + C$$

$$1 + 1 = 2$$