Math 122
Outline of Topics

I. Techniques of Integration

A. Substitution

1. $\int \left[f(u) \dfrac{du}{dx} \right] \, dx = \int f(u) \, du$
2. $u = g(x)$
3. $\frac{du}{dx} = g'(x)$

B. Parts

1. $\int u \, dv = uv - \int v \, du$
2. Choose u and dv
3. Find v and du

C. Trignometric Functions

1. $\sin^p x \cos^q x$
   (a) p is odd, let $u = \cos x$
   (b) q is odd, let $u = \sin x$
   (c) Both even, use double angle formulas.
2. $\tan^m x \sec^n x$
   (a) m is odd, let $u = \sec x$
   (b) n is even, let $u = \tan x$
   (c) $\int \tan x \, dx = \ln |\sec x| + C$
   (d) $\int \sec x \, dx = \ln |\sec x + \tan x| + C$

D. Trignometric Substitution

1. If the integral contains $\sqrt{a^2 - x^2}$, try $x = a \sin \theta$
2. If the integral contains $\sqrt{a^2 + x^2}$, try $x = a \tan \theta$
3. If the integral contains $\sqrt{x^2 - a^2}$, try $x = a \sec \theta$

E. Hyperbolic Functions

1. $\sinh x$
   (a) $y = \sinh x = \dfrac{e^x - e^{-x}}{2}$
   (b) $\dfrac{d}{dx} \sinh x = \cosh x$
   (c) $\displaystyle\int \sinh x \; dx = \cosh x + C$
2. $\cosh x$
   (a) $y = \sinh x = \dfrac{e^x + e^{-x}}{2}$
   (b) $\dfrac{d}{dx} \cosh x = \sinh x$
   (c) $\displaystyle\int \cosh x \; dx = \sinh x + C$
3. $\tanh x$
   (a) $y = \tanh x = \dfrac{\sinh x}{\cosh x}$
   (b) $\dfrac{d}{dx} \tanh x = \operatorname{sech}^2 x$
   (c) $\displaystyle\int \operatorname{sech}^2 x \; dx = \tanh x + C$

F. Partial Fractions

1. Non-repeating Linear Factors
2. Repeated Linear Factors
3. Quadratic Factors

G. Improper Integrals

1. $\int_a^{\infty} f(x)\ dx = \lim_{b\to\infty} \int_a^b f(x)\ dx$
2. $\int_{-\infty}^{b} f(x)\ dx = \lim_{a\to-\infty} \int_a^b f(x)\ dx$
3. $\int_a^b f(x)\ dx = \lim_{t\to b} \int_a^t f(x)\ dx$ if $f(x)$ is undefined at b.
4. $\int_a^b f(x)\ dx = \int_a^c f(x)\ dx + \int_c^b f(x)\ dx$ if $f(x)$ is undefined at c.

H. Numerical Methods

1. Trapizodial Method
   (a) $\Delta x = \frac{b-a}{n}$
   (b) $A = \frac{\Delta x}{2}\left(f(a) + 2f(x_1) + 2f(x_2) + \cdots + f(b)\right)$
2. Simpson's Method
   (a) $\Delta x = \frac{b-a}{n}$
   (b) $A = \frac{\Delta x}{3}\left(f(a) + 4f(x_1) + 2f(x_2) + 4f(x_3) + \cdots + f(b)\right)$

II. Applications of Integration

A. Probability

1. $p(x)$ is a probability density function if $p(x) \geq 0$ and $\int_S p(x)dx = 1$
2. $P(a \leq x \leq b) = \int_a^b p(x)\ dx$
3. $\mu = \int_S x\ p(x)dx$

B. Arclength, Surface Area

1. $S = \int_a^b \sqrt{1 + (f'(x))^2}\ dx$

2. S.A. $= \int_a^b 2\pi f(x)\sqrt{1 + (f'(x))^2}\ dx$

C. Fluid Force

1. $F = \int_a^b \rho\ h(x)w(x)\ dx$

D. Center of Mass

1. $\bar{x} = \dfrac{\int_a^b x\left[f(x) - g(x)\right]\ dx}{\int_a^b \left[f(x) - g(x)\right]\ dx}$

2. $\bar{y} = \dfrac{\int_a^b \dfrac{\left[f^2(x) - g^2(x)\right]}{2}\ dx}{\int_a^b \left[f(x) - g(x)\right]\ dx}$

E. Taylor Polynomials

1. $P_n(x) = \sum_{k=0}^{n} \dfrac{f^k(x_0)}{k!}(x - c)^k$

2. $R_n(x) = \dfrac{f^{n+1}(z)}{n+1)!}(x - c)^{n+1}$

III. Differential Equations

A. Separable

1. $y' = \frac{R(x)}{L(y)}$

2. $\int L(y)\ dy = \int R(x)\ dx$

B. $y' = k(y - b)$

1. $y = b + Ce^{kt}$
2. Newton's Law of Cooling

C. Slope Fields and Euler's Method

1. $x_{n+1} = x_n + h$
2. $y_{n+1} = y_n + h \cdot F(x_n, y_n)$

D. Logistic Equation

1. $\frac{dy}{dt} = ky\left(1 - \frac{y}{A}\right)$
2. $y = \frac{A}{1 - \frac{e^{-kt}}{C}} \qquad C = \frac{y_0}{y_0 - A}$
3. $A =$ carrying capacity

E. First order linear

1. $y' + P(x)y = Q(x)$
2. $\rho(x) = e^{\int P(x)\ dx}$
3. $y = \frac{1}{\rho(x)}\left[\int \rho(x)Q(x)\ dx + C\right]$

F. Applications

1. Exponential Growth and Decay
2. Rate in - rate out

IV. Sequences and Series

A. Sequences

1. Terms of a sequence
2. Limit of a sequence
3. Increasing, decreasing
4. Monotonic, Bounded

B. Definition of Convergence of a Series

1. $S_N = \sum_{n=1}^{N} a_n$
2. $S = \lim_{N \to \infty} S_N$

C. Geometric and Harmonic Series

1. Geometric: $\sum_{n=0}^{\infty} a\ r^n$
   (a) Converges if $|r| < 1$, diverges if $|r| \geq 1$
   (b) If it converges, the sum is $\frac{a}{1-r}$
2. Harmonic: $\sum_{n=1}^{\infty} \frac{1}{n}$
   (a) Diverges

D. Divergence Test

1. $\lim_{n \to \infty} a_n = L$
2. $L = 0$ tells you nothing
3. $L \neq 0$ tells you the series diverges

E. Integral Test

1. $a_n = f(n)$ for all n, $f'(x) < 0$
2. If $\int_1^\infty f(x)\,dx$ converges then $\sum_{n=1}^{\infty} a_n$ converges
3. If $\int_1^\infty f(x)\,dx$ diverges then $\sum_{n=1}^{\infty} a_n$ diverges

F. p-Test: $\sum_{n=1}^{\infty} \frac{1}{n^p}$

1. Converges if $p > 1$
2. Diverges if $p \leq 1$

G. Comparison Test and Limit Comparison Test

1. If $a_n < b_n$ and $\sum_{n=1}^{\infty} b_n$ converges then $\sum_{n=1}^{\infty} a_n$ converges
2. If $a_n > b_n$ and $\sum_{n=1}^{\infty} b_n$ diverges then $\sum_{n=1}^{\infty} a_n$ diverges
3. If $\lim_{n\to\infty} \frac{a_n}{b_n} = L$ and $0 < L < \infty$, then both $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ converge or both $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ diverge.

H. Ratio Test - Root Test

1. $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = L$ or $\lim_{n\to\infty} \sqrt[n]{a_n} = L$
2. If $L > 1$ the series diverges
3. If $L < 1$ the series converges
4. If $L = 1$ it tells you nothing

I. Alternating Series Test

1. The series alternates
2. $\lim_{n\to\infty} a_n = 0$
3. $|a_{n+1}| < |a_n|$

J. Power Series

1. Radius of Convergence
2. Interval of Convergence

K. Taylor Series

1. $f(x) = \sum_{n=0}^{\infty} \frac{f^n(x_0)}{n!}(x-c)^n$
2. Maclaurin Series $c = 0$

V. Parametric Equations

A. Slope and Concavity for Parametric Graphs

1. $\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$
2. $\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left[\frac{dy}{dx}\right]}{\frac{dx}{dt}}$

B. Polar Equations and Graphs

1. $x = r\cos\theta$
2. $y = r\sin\theta$
3. $r^2 = x^2 + y^2$

C. Area in Polar coordinates

1. $A = \int_{\alpha}^{\beta} \frac{f(\theta)^2}{2} d\theta$

D. Arc length

1. Parametric: $L = \int_{a}^{b} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$
2. Polar: $L = \int_{\alpha}^{\beta} \sqrt{r^2 + (r')^2}\, d\theta$

VI. Vectors

A. Algebra of Vectors

1. Addition and Subtraction
2. Scalar multiplication
3. Unit Vectors

B. Dot product

1. $\overrightarrow{A} \cdot \overrightarrow{B} = a_1 b_1 + a_2 b_2 + a_3 b_3$
2. $\overrightarrow{A} \cdot \overrightarrow{B} = ||\overrightarrow{A}||\ ||\overrightarrow{B}|| \cos\theta$
3. $\overrightarrow{A} \cdot \overrightarrow{B} = 0$ if and only if $\overrightarrow{A} \perp \overrightarrow{B}$

C. Cross Product

1. $\overrightarrow{A} \times \overrightarrow{B} = \langle a_2 b_3 - a_3 b_2, a_3 b_1 - a_1 b_3, a_1 b_2 - a_2 b_1 \rangle$
2. $||\overrightarrow{A} \times \overrightarrow{B}|| = ||\overrightarrow{A}||\ ||\overrightarrow{B}|| \sin\theta$
3. $\overrightarrow{A} \times \overrightarrow{B} = \overrightarrow{0}$ if and only if $\overrightarrow{A}$ and $\overrightarrow{B}$ go in the same direction.
4. $(\overrightarrow{A} \times \overrightarrow{B}) \perp \overrightarrow{A}$ and $(\overrightarrow{A} \times \overrightarrow{B}) \perp \overrightarrow{B}$

D. Lines

1. Point $P = (x_0, y_0, z_0)$
2. Direction Vector $\overrightarrow{V} = \langle a, b, c \rangle$
3. $x = x_0 + at, y = y_0 + bt, z = z_0 + ct$

E. Planes

1. Point $P = (x_0, y_0, z_0)$
2. Normal Vector $\overrightarrow{N} = \langle a, b, c \rangle$
3. $ax + by + cz = d$ where $d = ax_0 + by_0 + cz_0$