Math 122 Final

SI: ☐ **December 9, 2025** EF: ☐

1 - 2	/20
3 - 4	/20
5 - 6	/20
7 - 8	/20
9 - 10	/20
11- 12	/20
13 - 14	/20
15 - 16	/20
17 - 18	/20
19 - 20	/20
Total	/200

Name KEY

Directions:

1. No book, notes, or getting eaten by a dinosaur. You may use a calculator to do routine arithmetic computations. You may *not* use your calculator to store notes or formulas. You may not share a calculator with anyone.
2. You should show your work, and explain how you arrived at your answers. A correct answer with no work shown (except on problems which are completely trivial) will receive no credit. If you are not sure whether you have written enough, please ask.
3. You may not make more than one attempt at a problem. If you make several attempts, you must indicate which one you want counted, or you will be penalized.
4. You may leave as soon as you are finished, but once you leave the exam, you may not make any changes to your exam.

1. (10 points)

(a) Compute: $\int_0^1 (x+1)e^{x^2+2x}\,dx$.

$u = x^2 + 2x$

$du = 2x + 2\,dx$

$dx = \dfrac{du}{2x+2}$

$$= \int \cancel{(x+1)}\, e^u \frac{du}{\cancel{2x+2}\ 2} = \frac{1}{2} e^u = \frac{1}{2} e^{x^2+2x}\Big|_0^1$$

$$= \boxed{\frac{1}{2}(e^3 - 1)}$$

(b) If

$$\int f(x)\cos x\,dx = f(x)\sin x - \int x^5 \sin x\,dx$$

then what is $f(x)$?

$u = f(x) \quad dv = \cos x\,dx$

$du = f'(x)\,dx \quad v = \sin x$

$$= f(x)\sin x - \int f'(x)\sin x\,dx$$

$f'(x) = x^5$

$$\boxed{f(x) = \frac{x^6}{6} + C}$$

2. (10 points)

(a) Compute $\int (\cos^3 x)\sqrt{\sin x}\,dx$

$u = \sin x$

$du = \cos x\,dx$

$dx = \dfrac{du}{\cos x}$

$$= \int \cos^{\cancel{3}\,2} x \sqrt{u}\, \frac{du}{\cancel{\cos x}} = \int (1-u^2)\sqrt{u}\,du$$

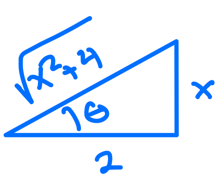
$$= \frac{2u^{3/2}}{3} - \frac{2u^{7/2}}{7} = \boxed{\frac{2}{3}(\sin x)^{3/2} - \frac{2}{7}(\sin x)^{7/2} + C}$$

(b) Compute $\int \frac{1}{\sqrt{x^2+4}}\,dx$

$x = 2\tan\theta$

$dx = 2\sec^2\theta\,d\theta$

$$= \int \frac{\cancel{2}\sec^{\cancel{2}}\theta}{\cancel{\sqrt{4\tan^2\theta + 4}}}\,d\theta = \ln|\sec\theta + \tan\theta|$$



$$= \boxed{\ln\left|\frac{\sqrt{x^2+4}}{2} + \frac{x}{2}\right| + C}$$

3. (10 points)

(a) Compute $\int \frac{\cos x}{\sin^2 x - 3\sin x + 2}\,dx$

$u = \sin x$

$du = \cos x\,dx$

$$= \int \frac{1}{u^2 - 3u + 2}\,du = \int \frac{1}{u-2} - \frac{1}{u-1}\,du = \ln|u-2| - \ln|u-1|$$

$$= \boxed{\ln|\sin x - 2| - \ln|\sin x - 1| + C}$$

$$\frac{1}{(u-2)(u-1)} = \frac{A}{u-2} + \frac{B}{u-1} = \frac{A(u-1) + B(u-2)}{(u-2)(u-1)}$$

$A = 1$

$B = -1$

(b) Compute $\int_0^\infty x e^{-x^2}\,dx$

$u = -x^2$ $\quad du = -2x\,dx$

$dx = \frac{du}{-2x}$

$$= \int \cancel{x}\, e^u \frac{du}{-2\cancel{x}} = -\frac{1}{2} e^u$$

$$= -\frac{1}{2} e^{-x^2} \Big|_0^\infty = \boxed{\frac{1}{2}}$$

4. (10 points) For the function $p(x) = \frac{C}{(x+2)^2}$ for $0 \leq x < \infty$.

(a) Find the value of C so the $p(x)$ is a probability density function.

$$\int_0^\infty \frac{C}{(x+2)^2}\,dx = -\frac{C}{x+2}\Big|_0^\infty = \frac{C}{2} = 1$$

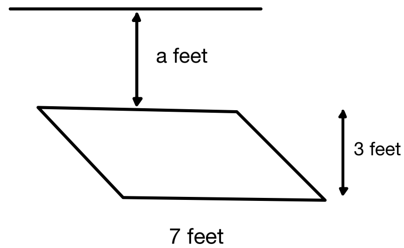
$$\boxed{C = 2}$$

(b) Find the $P(X \geq 1)$.

$$P(X \geq 1) = \int_1^\infty \frac{2}{(x+2)^2}\,dx = -\frac{2}{x+2}\Big|_1^\infty$$

$$= \boxed{\frac{2}{3}}$$

5. (10 points)A vertical flat plate in the shape of a parallelogram (see below) is submerged a feet below the water's surface. Determine the value of a if the force on the plate is 5,241.6 lbs. Assume that the top edge of the plate is parallel to the water's surface. ($\rho g = 62.4$ lb/ft^3)



$$F = \int_0^3 (62.4)(x+a)(7)\,dx$$

$$= (62.4)(7)\left[\frac{x^2}{2} + ax\right]_0^3$$

$$= (62.4)(7)\left(\frac{9}{2} + 3a\right) = 5241.6$$

$$\frac{9}{2} + 3a = 12$$

$$\boxed{a = 2.5 \text{ ft}}$$

6. (10 points) Find the length of the curve $f(x) = \cosh x$ from $[0, \ln 2]$.

$f(x) = \cosh x \qquad f'(x) = \sinh x$

$$S = \int_0^{\ln 2} \sqrt{1 + \sinh^2 x}\,dx = \int_0^{\ln 2} \cosh x\,dx$$

$$= \sinh x \Big|_0^{\ln 2} = \sinh(\ln 2) = \boxed{\frac{3}{4}}$$

7. (10 points) Find the solution of

$$y' + (\tan x)\, y = \sec x \qquad y(0) = 2$$

$P(x) = \tan x$ $\quad Q(x) = \sec x$ $\quad \alpha(x) = e^{\int \tan x \, dx} = e^{-\ln \cos x} = (\cos x)^{-1} = \sec x$

$$y = \frac{1}{\sec x}\left[\int \sec^2 x \, dx + c\right]$$

$$y = \cos x \left[\tan x + c\right]$$

$$2 = 1\left[0 + c\right] \qquad c = 2$$

$$\boxed{y = \cos x \left[\tan x + 2\right]}$$

8. (10 points) The population of cockroaches in the Sears basement, with a carrying capacity of 5,000, and grow constant 0.2, follows the logistic growth model:

$$y' = ky\left(1 - \frac{y}{A}\right)$$

Initially, there are 500 cockroaches.

Find:

(a) $y(0) = 500$

(b) $A = 5000$

(c) $B = \dfrac{y_0}{y_0 - A} = \dfrac{500}{500 - 5000} = -\dfrac{1}{9}$

(d) $k = .2$

(e) Solve the differential equation for find $y(t)$

$$y(t) = \frac{5000}{1 + 9e^{-.2t}}$$

(f) How many cockroaches will there be after 5 years?

$$y(5) = \frac{5000}{1 + 9e^{-1}} = 1160 \text{ COCKROACHES}$$

9. (10 points) A tank contains 30 gallons of black coffee and 1 pound of sugar. A flavored coffee syrup containing 1.5 pounds of sugar per gallon flows into the tank at 4 gallons per minute. The well-stirred mixture drains out at 4 gallons per minute.

(a) Write a differential equation (with initial condition) for the amount of sugar in the tank at any time.

$$\frac{dy}{dt} = R_1 - R_0 = (1.5)(4) - 4\left(\frac{y}{30}\right) = \boxed{6 - \frac{2}{15}y \quad y(0)=1}$$

(b) Solve the differential equation from part (a).

$$P(t) = \frac{2}{15} \qquad Q(t) = 6 \qquad \alpha(t) = e^{\int \frac{2}{15}dt} = e^{\frac{2}{15}t}$$

$$y = \frac{1}{e^{\frac{2}{15}t}}\left[\int 6e^{\frac{2}{15}t}\,dt + C\right] = e^{-\frac{2}{15}t}\left[45e^{\frac{2}{15}t} + C\right]$$

$$\boxed{y = 45 - 44e^{-\frac{2}{15}t}}$$

(c) How many pounds of sugar will be in the tank after 8 minutes?

$$y(8) = 45 - 44e^{-\frac{2}{15}(8)}$$

$$= \boxed{29.85 \text{ Lbs}}$$

10. (10 points)

(a) Find the limit of the sequence, assuming it exists.

$$a_1 = \sqrt{30} \qquad a_{n+1} = \sqrt{30 + a_n}$$

$$\lim_{N\to\infty} a_{N+1} = \lim_{N\to\infty} \sqrt{30 + a_N}$$

$$L = \sqrt{30 + L} \qquad L^2 = 30 + L$$

$$L^2 - L - 30 = 0$$

$$(L-6)(L+5) = 0 \qquad \boxed{L = 6}, \cancel{-5}$$

(b) Find the sum of the series: $\displaystyle\sum_{n=0}^{\infty} \frac{3^{n+1}}{2^{2n+1}}$

$$= \sum_{N=0}^{\infty} \frac{3\cdot 3^N}{2\cdot 4^N} = \sum_{N=0}^{\infty} \frac{3}{2}\left(\frac{3}{4}\right)^N$$

$$= \frac{\frac{3}{2}}{1-\frac{3}{4}} = \frac{\frac{3}{2}}{\frac{1}{4}} = \boxed{6}$$

11. (10 points) Determine if the following series converge or diverge. For each test you use, you must name the test, perform the test, and state the conclusion you reached from that test.

(a) $\sum_{n=1}^{\infty} \frac{1}{2+3^{-n}}$ N-TH TERM

$\lim_{N\to\infty} \frac{1}{2+3^{-N}} = \frac{1}{2} \neq 0$ $\boxed{\text{DIV}}$

(b) $\sum_{n=1}^{\infty} \left(\frac{2n+1}{5+3n}\right)^n$ ROOT TEST

$\lim_{N\to\infty} \sqrt[N]{\left(\frac{2N+1}{5+3N}\right)^N} = \lim_{N\to\infty} \frac{2N+1}{5+3N} = \frac{2}{3} < 1$

$\boxed{\text{CONV}}$

12. (10 points) Determine if the following series converge absolutely, converge conditionally, or diverge. For each test you use, you must name the test, perform the test, and state the conclusion you reached from that test.

(a) $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{\ln(n+1)}{n}$ COMP TEST $\frac{\ln(N+1)}{N} > \frac{1}{N}$ $\sum \frac{1}{N}$ DIV.

AST

$\lim_{N\to\infty} \frac{\ln(N+1)}{N} = 0$ ✓ $\frac{\ln(N+2)}{N+2} < \frac{\ln(N+1)}{N}$ ✓

$\boxed{\text{CONV. COND}}$

(b) $\sum_{n=1}^{\infty} (-1)^n \left(\frac{n^2}{3^n}\right)$ RATIO TEST

$\lim_{N\to\infty} \frac{(N+1)^2}{3^{\cancel{N+1}}} \frac{3^{\cancel{N}}}{N^2} = \frac{1}{3} < 1$

$\boxed{\text{CONV. ABS}}$

13. (10 points) Consider the power series:

$$\sum_{n=1}^{\infty} n3^n(x-2)^n$$

(a) Where is the power series centered?

$\boxed{C = 2}$

(b) Find the radius of convergence.

$$\lim_{N\to\infty} \left| \frac{(N+1)\, \cancel{3^{N+1}}\, \cancel{(x-2)^{N+1}}}{N\, \cancel{3^N}\, \cancel{(x-2)^N}} \right| = |3(x-2)| < 1$$

$$|x-2| < \frac{1}{3}$$

$\boxed{R = \frac{1}{3}}$

(c) Find the interval of convergence. [Hint: Check the endpoints.]

$x = \frac{7}{3}$ $\quad \sum_{N=1}^{\infty} N 3^N \left(\frac{7}{3} - 2\right)^N = \sum_{N=1}^{\infty} N$ DIV

$x = \frac{5}{3}$ $\quad \sum_{N=1}^{\infty} N 3^N \left(\frac{5}{3} - 2\right)^N = \sum_{N=1}^{\infty} (-1)^N N$ DIV

$\boxed{\left(\frac{5}{3}, \frac{7}{3}\right)}$

14. (10 points) The Maclaurin series for $f(x) = \ln(1+x^2)$ is

$$\ln(1+x^2) = x^2 - \frac{x^4}{2} + \frac{x^6}{3} - \frac{x^8}{4} + \frac{x^{10}}{5} + \ldots$$

(a) Write the Maclaurin series for $x^2 f(x)$.

$$x^2 f(x) = x^4 - \frac{x^6}{2} + \frac{x^8}{3} - \frac{x^{10}}{4} + \frac{x^{12}}{5} \ldots$$

(b) Write the Maclaurin series for $g(x) = \dfrac{1}{1+x^2}$.

$$\frac{2x}{1+x^2} = 2x - \frac{4x^3}{2} + \frac{6x^5}{3} - \frac{8x^7}{4} + \frac{10x^9}{5} \ldots$$

$$\frac{1}{1+x^2} = 1 - \frac{2x^2}{2} + \frac{3x^4}{3} - \frac{4x^6}{4} \ldots = \boxed{1 - x^2 + x^4 - x^6 \ldots}$$

(c) Find $T_4(x)$ the fourth degree Maclaurin polynomial for $f(x)$.

$$\boxed{T_4(x) = x^2 - \frac{x^4}{2}}$$

15. (10 points) Consider the parametric curve given by

$$c(t) = (\cos^2 t, \sin^2 t) \qquad c\left(\frac{\pi}{4}\right) = \left(\frac{1}{2}, \frac{1}{2}\right)$$

(a) Find the equation of the tangent line at $t = \frac{\pi}{4}$.

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{2\sin t \cos t}{-2\cos t \sin t} = -1$$

$$y - \frac{1}{2} = -\left(x - \frac{1}{2}\right)$$

$$\boxed{y = -x + 1}$$

(b) Find the length of the curve for $0 \leq t \leq \frac{\pi}{2}$.

$$\left(\frac{dx}{dt}\right)^2 = 4\cos^2 t \sin^2 t \qquad \left(\frac{dy}{dt}\right)^2 = 4\sin^2 t \cos^2 t$$

$$S = \int_0^{\frac{\pi}{2}} \sqrt{4\cos^2 t \sin^2 t + 4\sin^2 t \cos^2 t}\, dt$$

$$= \int_0^{\frac{\pi}{2}} 2\sqrt{2} \cos t \sin t \, dt = \sqrt{2} \sin^2 t \Big|_0^{\frac{\pi}{2}}$$

$$= \boxed{\sqrt{2}}$$

16. (10 points) Find the area inside $r = 3\sin(n\theta)$ if n is an odd integer.

$$A = \int_0^{\pi} \frac{(3\sin(n\theta))^2}{2} \, d\theta$$

$$= \frac{9}{2} \int_0^{\frac{\pi}{2}} \sin^2(n\theta)\, d\theta = \frac{9}{2} \int_0^{\pi} \frac{1 - \cos 2n\theta}{2} \, d\theta$$

$$= \frac{9}{4} \left[\theta - \frac{\sin 2n\theta}{2} \right]_0^{\pi}$$

$$= \boxed{\frac{9\pi}{4}}$$

17. (10 points) Let $\overrightarrow{a} = \langle 4, 2, 1 \rangle$ and $\overrightarrow{b} = \langle 5, 1, -1 \rangle$ find

(a) $\vec{a} \cdot \vec{b}$.

$4(5) + 2(1) + 1(-1) = \boxed{21}$

(b) $\cos\theta$ where θ is the angle between $\overrightarrow{a}$ and $\overrightarrow{b}$.

$\cos\theta = \dfrac{\vec{a} \cdot \vec{b}}{\|\vec{a}\| \|\vec{b}\|} = \boxed{\dfrac{21}{\sqrt{21}\sqrt{27}}}$

(c) $||\text{proj}_{\overrightarrow{a}} \overrightarrow{b}||$

$= \dfrac{|\vec{a} \cdot \vec{b}|}{\|\vec{a}\|} = \dfrac{21}{\sqrt{21}} = \boxed{\sqrt{21}}$

(d) $\vec{a} \times \vec{b}$.

$$\begin{matrix} i & j & k & i & j \\ 4 & 2 & 1 & 4 & 2 \\ 5 & 1 & -1 & 5 & 1 \end{matrix}$$

$\boxed{\vec{a} \times \vec{b} = \langle -3, 9, -6 \rangle}$

(e) the area of the triangle with vertices $(2, 1, 1)$, $(6, 3, 2)$ and $(7, 2, 0)$
P Q R

$\vec{PQ} = \langle 4, 2, 1 \rangle$

$\vec{PR} = \langle 5, 1, -1 \rangle$

$\vec{PQ} \times \vec{PR} = \langle -3, 9, -6 \rangle$

$\text{AREA} = \frac{1}{2}\sqrt{9 + 81 + 36} = \boxed{\dfrac{\sqrt{126}}{2}}$

18. (10 points)

(a) Are the line $x = 1 + t, y = 2 - 5t, z = 3 + 2t$ and the plane $2x + 2y + 4z = 8$ parallel, perpendicular or neither (and why).

$\vec{V} = \langle 1, -5, 2 \rangle \qquad \vec{N} = \langle 2, 2, 4 \rangle$

$\vec{V} \cdot \vec{N} = 2 - 10 + 8 = 0$

$\vec{V} \perp \vec{N}$ SO $\boxed{\text{LINE IS PARALLEL TO PLANE}}$

(b) The lines two lines

$$L_1 : x = 1 + 2t, \qquad y = 3t, \qquad z = 2 - t$$

$$L_2 : x = 9 + 2t, \qquad y = 5 - 4t, \qquad z = 1 + 2t$$

intersect at the point (a, b, c). What is b?

$1 + 2t = 9 + 2s$

$3t = 5 - 4s$

$2 + 4t = 18 + 4s$

$2 + 7t = 23$

$7t = 21$

$t = 3 \Rightarrow s = -1$

$t = 3 \Rightarrow (7, 9, -1)$

$s = -1 \Rightarrow (7, 9, -1)$

$\boxed{b = 9}$

19. (10 points)

(a) Find the point of intersection of the plane $3x - 2y + 2z = 7$ and the line $x = 1 - 2t, y = 9, z = 1 + t$

$3(1-2t) - 2(9) + 2(1+t) = 7$

$3 - 6t - 18 + 2 + 2t = 7$

$4t = -20 \quad t = -5$ $\boxed{\text{POINT } (11, 9, -4)}$

(b) Find the equation of the plane that contains the points $(2, 4, 1)$, $(2, 3, 2)$, and $(3, -2, 6)$.

$\vec{PQ} = \langle 0, 1, -1 \rangle \qquad \vec{PQ} \times \vec{PR} = \langle -1, -1, -1 \rangle$

$\vec{PR} = \langle 1, -6, 5 \rangle$

$\boxed{x + y + z = 7}$

20. (10 points)

(a) Find the distance from the point (2, 4, -3) to the plane

$$x + 2y - 4z = 1$$

$$\text{DIST} = \frac{|2 + 2(4) + 12 - 1|}{\sqrt{1+4+16}} = \frac{21}{\sqrt{21}} = \boxed{\sqrt{21}}$$

(b) Find the equation of the plane that contains the point $(1, -1, 2)$ and is perpendicular to the two planes

$$x + y + z = 3 \qquad x + 2y - z = 4$$

$$\begin{array}{lllll} & i & j & k & \\ \vec{N}_1 = & \langle 1, & 1, & 1 \rangle & 1\;1 \\ \vec{N}_2 = & \langle 1, & 2, & -1 \rangle & 1\;2 \end{array} \qquad \vec{N}_1 \times \vec{N}_2 = \langle -3, 2, 1 \rangle$$

$\boxed{-3x + 2y + z = -3}$