# Math 122 Test 3

April 15, 2014

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Directions:

1. No books, notes or 6 year olds with ear infections. You may use a calculator to do routine arithmetic computations. You may *not* use your calculator to store notes or formulas. You may not share a calculator with anyone.

2. You should show your work, and explain how you arrived at your answers. A correct answer with no work shown (except on problems which are completely trivial) will receive no credit. If you are not sure whether you have written enough, please ask.

3. You may not make more than one attempt at a problem. If you make several attempts, you must indicate which one you want counted, or you will be penalized.

4. Numerical experiments do not count as justification. For example, computing the first few terms of a series is not enough to show the terms decrease. Computing the first few partial sums of a series is not enough to show the series converges or diverges. Plugging in numbers is not enough to justify the computation of a limit. You may, of course use numerical experiments to help guide your work, but they do not count as justification for answers.

5. On this test, explanations count. If I can't follow what you are doing, you will not get much credit.

6. You may leave as soon as you are finished, but once you leave the exam, you may not make any changes to your exam.
1. (10 points)

(a) Compute the limit of the following sequences:

\[
\lim_{n \to \infty} \left\{ n (\sqrt{n^2 + 1} - n) \right\}^\infty_{n=1}
\]

\[
= \lim_{N \to \infty} \frac{N \left(\sqrt{N^2 + 1} - N\right)}{\left(\sqrt{N^2 + 1} + N\right)}
\]

\[
= \lim_{N \to \infty} \frac{N \left(\sqrt{N^2 + 1} - N\right)}{\sqrt{N^2 + 1} + N} = \frac{1}{2}
\]

(b) Determine if the following series converges or diverges, and if it converges, find the sum:

\[
\sum_{n=0}^{\infty} 5 \left(\frac{-2}{3}\right)^n
\]

CONV.

GEO

\[
R = -\frac{2}{3}, \quad \left| -\frac{2}{3} \right| < 1
\]

\[
S = \frac{5}{1 - \left(-\frac{2}{3}\right)} = \frac{5}{\frac{5}{3}} = 3
\]
2. (15 points) For each of the following series, determine if it converges or diverges. For each test you use, you must name the test, perform the test, and state the conclusion you reached from that test.

(a) \[ \sum_{n=1}^{\infty} \frac{n^3}{n^5 + 10n^4 + 3n^3 + 1} \]  
\text{CONV BY COMP TEST} 

\[ \frac{N^3}{N^5 + 10N^4 + 3N^3 + 1} < \frac{1}{N^2} \leq \frac{1}{N^2} \text{ CONV} \]

(b) \[ \sum_{n=1}^{\infty} \frac{(n!)^2 2^n}{(2n)!} \]  
\text{CONV. BY RATIO TEST} 

\[ \lim_{N \to \infty} \frac{((N+1)!)^2 2^{N+1}}{(2N+2)!} \frac{2N!}{(N!)^2 2^N} \]

\[ = \lim_{N \to \infty} \frac{a N+1}{2N+1} = \frac{1}{2} < 1 \]

(c) \[ \sum_{n=1}^{\infty} \frac{n^n}{(3n+5)^n} \]  
\text{CONV. BY ROOT TEST} 

\[ \lim_{N \to \infty} \frac{N^n}{(3N+5)^n} = \]

\[ = \lim_{N \to \infty} \frac{N}{3N+5} = \frac{1}{3} < 1 \]
3. (15 points) Determine if the following series converge absolutely, converge conditionally, or diverge. For each test you use, you must name the test, perform the test, and state the conclusion you reached from that test.

(a) \[ \sum_{n=1}^{\infty} \frac{(-1)^n(n^2 + 1)}{2n^3 + n - 1} \] 

DIV. BY DIV. TEST

\[ \lim_{N \to \infty} \frac{N^2 + 1}{2N^2 + N - 1} = \frac{1}{2} \neq 0 \]

(b) \[ \sum_{n=2}^{\infty} \frac{(-1)^n}{n(n \ln n)^2} \]

LOOK AT \[ \frac{1}{N(\ln N)^2} \]

CONV. ABS

INTEGRAL TEST

\[ f(x) = \frac{1}{x(\ln x)^2} \]

CONT \checkmark DECR \checkmark

\[ \int_{2}^{\infty} \frac{1}{x(\ln x)^2} \, dx \]

\[ = \int \frac{1}{u^2} \, du = -\frac{1}{u} \]

\[ = -\frac{1}{\ln x} \bigg|_{2}^{\infty} = \frac{1}{\ln 2} \]

(c) \[ \sum_{n=2}^{\infty} \frac{(-1)^n}{\sqrt{n} + 1} \]

L.C.T. \[ \frac{1}{\sqrt{N}} \]

\[ \lim_{N \to \infty} \frac{\sqrt{N}}{\sqrt{N+1}} = \lim_{N \to \infty} \frac{\sqrt{N}^2 + 1}{\sqrt{N}} = 1 \]

GOTO AST.

A.S.T \[ \lim_{N \to \infty} \frac{1}{\sqrt{N+1}} = 0 \]

\[ \frac{1}{\sqrt{N+1}} < \frac{1}{\sqrt{N} + 1} \]

CONV. COND.
4. (10 points) Consider the power series:

\[ \sum_{n=2}^{\infty} \frac{(x-3)^n}{2^n \ln n} \]

(a) Find the radius of convergence.

\[
\lim_{N \to \infty} \left| \frac{(x-3)^{N+1}}{2^{N+1} \ln(N+1)} \frac{\ln N}{(x-3)^N} \right| = \left| \frac{x-3}{2} \right| < 1
\]

\[
|x-3| < 2 \quad \therefore R = 2
\]

(b) Find the interval of convergence. [Hint: Check the endpoints.]

\[ x = 5 \]

\[ \sum_{n=2}^{\infty} \frac{5^n}{2^n \ln n} = \sum_{n=2}^{\infty} \frac{1}{\ln n} > \frac{1}{2} \quad \sum_{n=2}^{\infty} \frac{1}{n} \quad \text{div.} \]

\[ x = 1 \]

\[ \sum_{n=2}^{\infty} \frac{(-1)^n}{2^n \ln n} = \sum_{n=2}^{\infty} \frac{1}{\ln n} \quad \text{AST} \quad \lim_{n \to \infty} \frac{1}{\ln n} = 0 \quad \checkmark \]

5. (10 points)

(a) Find the Maclaurin Series for \( f(x) = \frac{x}{(1-x)^2} \)

Hint: \( \frac{d}{dx} \left( \frac{1}{1-x} \right) = \frac{1}{(1-x)^2} \)

\[ \frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + x^5 + \ldots \]

\[ \frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + 4x^3 + 5x^4 + \ldots \]

\[ \frac{x}{(1-x)^2} = x + 2x^2 + 3x^3 + 4x^4 + \ldots \]

(b) Find \( \sum_{n=1}^{\infty} \frac{n}{2^n} \)

Hint: Let \( x = \frac{1}{2} \)

\[ \frac{x}{(1-x)^2} = \frac{x}{1/2} = 1 + 2x^2 + 3x^3 + 4x^4 + 5x^5 + \ldots \]

\[ \left. \frac{1}{(1-x)^2} = 2 = \frac{1}{2} + \frac{2}{2^2} + \frac{3}{2^3} + \frac{4}{2^4} + \ldots = \sum_{n=1}^{\infty} \frac{2^n}{2^n} \right|_{x=1/2} \]
6. (15 points) For the curve given parametrically by

\[ x = \cos^3 t \quad y = \sin^3 t \]

(a) Find \( \frac{dy}{dx} \) at \( t = \frac{\pi}{4} \).

\[
\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{3 \sin^2 t \cos t}{-3 \cos^2 t \sin t} \bigg|_{t = \frac{\pi}{4}} = -1
\]

(b) Find the equation of the tangent line at \( t = \frac{\pi}{4} \).

\[
X = \cos^3 \left( \frac{\pi}{4} \right) = \left( \frac{\sqrt{2}}{2} \right)^3 = \frac{\sqrt{2}}{8} = \frac{\sqrt{2}}{4}
\]

\[
Y = \sin^3 \left( \frac{\pi}{4} \right) = \left( \frac{\sqrt{2}}{2} \right)^3 = \frac{\sqrt{2}}{4}
\]

\[
Y - \frac{\sqrt{2}}{4} = -1 \left( X - \frac{\sqrt{2}}{4} \right)
\]

(c) Find the length of the curve for \( 0 \leq t \leq \frac{\pi}{2} \).

\[
\left( \frac{dx}{dt} \right)^2 = 9 \sin^4 t \cos^2 t \quad \left( \frac{dy}{dt} \right)^2 = 9 \cos^4 t \sin^2 t
\]

\[
S = \int_{0}^{\frac{\pi}{2}} \sqrt{9 \sin^2 t \cos^2 t \left( \sin^2 t + \cos^2 t \right)} \, dt
\]

\[
= \int_{0}^{\frac{\pi}{2}} 3 \sin t \cos t \, dt \quad u = \sin t \quad du = \cos t \, dt
\]

\[
= \int_{0}^{\frac{\pi}{2}} 3u^2 \, du = \frac{3u^3}{2} \bigg|_{0}^{\frac{\pi}{2}} = \frac{3}{2}
\]
7. (15 points)

(a) Sketch a graph of the area of the region inside both \( r = \sqrt{3} \cos \theta \) and \( r = \sin \theta \).

\[ \sin \theta = \sqrt{3} \cos \theta \]
\[ \tan \theta = \sqrt{3} \]
\[ \theta = \frac{\pi}{3} \]

(b) For the area of region inside both \( r = \sqrt{3} \cos \theta \) and \( r = \sin \theta \).

Fill in the boxes.

\[ A = \int_{0}^{\frac{\pi}{3}} \frac{(\sin \theta)^2}{2} d\theta + \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \frac{(\sqrt{3} \cos \theta)^2}{2} d\theta \]
8. (10 points)

(a) Find the equation of the ellipse obtained by translating the ellipse:

\[
\left(\frac{x - 8}{6}\right)^2 + \left(\frac{y + 4}{3}\right)^2 = 1
\]

so the center is at the origin.

\[
\left(\frac{x}{6}\right)^2 + \left(\frac{y}{3}\right)^2 = 1
\]

(b) Find the equation of the parabola that has vertex at (3, 1) and focus at (3, 4).

\[
y - 1 = \frac{1}{12} (x - 3)^2
\]