**Math 122 Test 3**  
November 22, 2016

<p>| | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td></td>
</tr>
<tr>
<td><strong>Total</strong></td>
<td>[KEY]</td>
</tr>
</tbody>
</table>

Directions:

1. No books, notes or going from summer to winter in one day. You may use a calculator to do routine arithmetic computations. You may *not* use your calculator to store notes or formulas. You may not share a calculator with anyone.

2. You should show your work, and explain how you arrived at your answers. A correct answer with no work shown (except on problems which are completely trivial) will receive no credit. If you are not sure whether you have written enough, please ask.

3. You may not make more than one attempt at a problem. If you make several attempts, you must indicate which one you want counted, or you will be penalized.

4. Numerical experiments do not count as justification. For example, computing the first few terms of a series is not enough to show the terms decrease. Computing the first few partial sums of a series is not enough to show the series converges or diverges. Plugging in numbers is not enough to justify the computation of a limit.

5. On this test, explanations count. If I can't follow what you are doing, you will not get much credit.

6. You may leave as soon as you are finished, but once you leave the exam, you may not make any changes to your exam.

**Happy Thanksgiving**
1. (20 points)

(a) Find the limit of the following sequences:

i. \( \left\{ \arctan\left(1 - \frac{5}{n}\right) \right\}_{n=1}^{\infty} \)

\[
\lim_{N \to \infty} \arctan\left(1 - \frac{5}{N}\right) = \arctan\left(1\right) = \frac{\pi}{4}
\]

ii. \( \left\{ n^2 \sin\left(\frac{1}{n^2}\right) \right\}_{n=1}^{\infty} \)

\[
\lim_{N \to \infty} \frac{\sin\left(\frac{1}{N^2}\right)}{\frac{1}{N^2}} = 1
\]

(b) Determine if the series

\[
\sum_{n=1}^{\infty} \frac{1}{n^2 + 2n}
\]

converges or diverges, and if it converges, find the sum.

\[
\frac{1}{N^2 + 2N} = \frac{A}{N} + \frac{B}{N + 2} = \frac{A(N + 2) + BN}{N(N + 2)}
\]

\[
A = \frac{1}{2}, \quad B = -\frac{1}{2}
\]

\[
= \frac{1}{2} \sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n + 2}\right)
\]

\[
= \frac{1}{2} \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \left(\frac{1}{4} - \frac{1}{5}\right) + \cdots
\]

\[
= \frac{1}{2} \left(\frac{3}{2}\right) = \frac{3}{4}
\]
2. (20 points)

(a) For each of the following series, determine if it converges or diverges. For each test you use, you must name the test, perform the test, and state the conclusion you reached from that test.

i. \( \sum_{n=1}^{\infty} \left( \frac{n+1}{n^2 + 2} \right)^n \) \[\text{CONV}\]

\[\text{ROOT TEST} \quad \lim_{N \to \infty} N \left( \frac{N+1}{N^2 + 2} \right)^N = \lim_{N \to \infty} \frac{N+1}{N^2 + 2} = 0\]

ii. \( \sum_{n=1}^{\infty} \sin \left( \frac{1}{n^3} \right) \) \[\text{CONV}\]

\[\text{L.C.T} \quad \lim_{N \to \infty} \frac{\sin \left( \frac{1}{N^3} \right)}{\frac{1}{N^2}} = 1 \quad \leq \frac{1}{N^2} \quad \text{CONV}\]

(b) For each of the following series, determine if the following series converge absolutely, converge conditionally, or diverge. For each test you use, you must name the test, perform the test, and state the conclusion you reached from that test.

i. \( \sum_{n=2}^{\infty} \frac{(-1)^n (n!)^2}{(2n+1)!} \) \[\text{CONV ABS}\]

\[\text{RATIO TEST} \quad \lim_{N \to \infty} \frac{\left( \frac{(N+1)!}{(2N+3)!} \right)^2 \left( \frac{(2N+1)!}{(N!)^2} \right) = \lim_{N \to \infty} \frac{(N+1)^2}{(2N+2)(2N+3)} = \frac{1}{4} < 1\]

ii. \( \sum_{n=1}^{\infty} \frac{(-1)^n \sin(2^n)}{n^{3/2}} \) \[\text{CONV. ABS}\]

\[\text{COMP TEST} \quad \frac{\sin (2^n)}{n^{3/2}} < \frac{1}{n^{3/2}} \quad \leq \frac{1}{n^{3/2}} \quad \text{CONV}\]

\[p = \frac{3}{2} > 1\]
3. (20 points)

(a) Consider the power series:

\[ \sum_{n=1}^{\infty} \frac{n^2(x - 3)^n}{5^n} \]

i. Where is the power series centered?

\[ C = 3 \quad -2, \quad 8 \]

ii. Find the radius of convergence.

\[ \lim_{N \to \infty} \left| \frac{(n+1)^2 (x-3)^{n+1}}{5^{n+1}} \cdot \frac{5^n}{n^2 (x-3)^n} \right| = \left| \frac{(x-3)}{5} \right| < 1 \]

\[ |x-3| < 5 \quad R = 5 \]

iii. Find the interval of convergence. [Hint: Check the endpoints.]

\[ x = 8 \quad N = 8 \quad \frac{n^2}{5^n} \quad \text{DIV} \]

\[ x = -2 \quad N = 8 \quad \frac{n^2}{5^n} \quad \text{DIV} \quad \text{TEST} \]

\[ (-2, \quad 8) \]

(b) i. Write out the first four non-zero terms for the Maclaurin series for \( f(x) = x^3 \sin(x^2) \)

\[ \sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots \]

\[ \sin (x^2) = x^2 - \frac{x^6}{3!} + \frac{x^{10}}{5!} - \frac{x^{14}}{7!} + \ldots \]

\[ x^3 \sin (x^2) = x^5 - \frac{x^9}{3!} + \frac{x^{13}}{5!} - \frac{x^{17}}{7!} + \ldots \]

ii. Find \( f^{(13)}(0) \) (the 13th derivative of \( f(x) \) at \( x = 0 \))

\[ \frac{1}{13!} = \frac{f^{(13)}(0)}{13!} \]

\[ f^{(13)}(0) = \frac{13!}{15!} \]

\[ = 259, \quad 459, \quad 200 \]
4. (20 points) For the curve \[ c(t) = (t - \frac{1}{3}t^3, t^2) \]

(a) Find \( \frac{dy}{dx} \).

\[
\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{2t}{1 - t^2}
\]

(b) Find the equation of the tangent line at \( t = 2 \).

\[ \left( -\frac{2}{3}, 4 \right) \]

\[
\left. \frac{dy}{dx} \right|_{t=2} = \frac{4}{-3}
\]

\[ y - 4 = \frac{4}{3} (x + \frac{2}{3}) \]

(c) The speed at \( t = 1 \).

\[
\frac{ds}{dt} = \sqrt{\left(1 - t^3\right)^2 + \left(2t^2\right)^2}
\]

\[
\left. \frac{ds}{dt} \right|_{t=1} = 2
\]

(d) The arc length for \( 0 \leq t \leq \sqrt{3} \).

\[
S = \int_{0}^{\sqrt{3}} \sqrt{1 - 2t^2 + t^4 + 4t^2} \, dt
\]

\[
= \int_{0}^{\sqrt{3}} \sqrt{1 + 2t^2 + t^4} \, dt = \int_{0}^{\sqrt{3}} \sqrt{(t + \frac{\sqrt{3}}{3})^2} \, dt
\]

\[
= \int_{0}^{\sqrt{3}} \left( t + \frac{\sqrt{3}}{3} \right) \, dt = \sqrt{3} + \frac{\sqrt{3} \cdot \sqrt{3}}{3} = 2\sqrt{3}
\]
5. (20 points)

(a) For the area of region inside \( r = \sin \theta \) and outside \( r = 1 - \cos \theta \).

Fill in the boxes.

\[
A = \frac{\pi}{2} \left( \frac{\text{SINE}}{2} \right)^2 d\theta - \frac{\pi}{2} \left( \frac{1-\text{COS}}{2} \right)^2 d\theta
\]

(b) For the ellipse

\[4x^2 + 9y^2 + 24x - 72y + 144 = 0\]

find the center, foci, and vertices.

\[4(x^2 + 6x + 9) + 9(y^2 - 8y + 16) = -144\]

\[+ 36\theta + 144\]

\[4(x + 3)^2 + 9(y - 4)^2 = 3c\]

\[\frac{(x+3)^2}{9} + \frac{(y-4)^2}{4} = 1\]

\[c^2 = a^2 - b^2 = 5\]

\[c = \sqrt{5}\]

\[\text{Center: } (-3, 4)\]

\[\text{Vertices: } (0, 4), (-6, 4)\]

\[(-3, 2), (-3, 6)\]

\[\text{Foci: } (-3 \pm \sqrt{5}, 4)\]