

SI:

Math 122 Test 2

March 3, 2026

EF:

Name KEY

1	
2	
3	
4	
5	
Total	

Directions:

1. No books, notes or letting Jonathan write the test. You may use a calculator to do routine arithmetic computations. You may *not* use your calculator to store notes or formulas. You may not share a calculator with anyone.
2. You should show your work, and explain how you arrived at your answers. A correct answer with no work shown (except on problems which are completely trivial) will receive no credit. If you are not sure whether you have written enough, please ask.
3. You may not make more than one attempt at a problem. If you make several attempts, you must indicate which one you want counted, or you will be penalized.
4. You may leave as soon as you are finished, but once you leave the exam, you may not make any changes to your exam.

1. (20 points)

(a) Let $p(x) = \frac{x+a}{16}$ for $0 \leq x \leq 4$

i. Find the value of a so that $p(x)$ is a probability density function.

$$\int_0^4 \frac{x+a}{16} dx = \frac{1}{16} \left(\frac{x^2}{2} + ax \right) \Big|_0^4 =$$
$$\frac{1}{16} (8 + 4a) = 1 \quad \boxed{a = 2}$$

ii. Find the mean μ for $p(x)$

$$N = \int_0^4 x \left(\frac{x+2}{16} \right) dx = \frac{1}{16} \left[\frac{x^3}{3} + x^2 \right]_0^4$$
$$= \frac{1}{16} \left[\frac{64}{3} + 16 \right] = \frac{4}{3} + 1 = \boxed{\frac{7}{3}}$$

(b) Find the arc length of the curve

$$f(x) = \arcsin x + \sqrt{1-x^2}$$

for $0 \leq x \leq 1$.

$$f'(x) = \frac{1}{\sqrt{1-x^2}} - \frac{x}{\sqrt{1-x^2}} = \frac{1-x}{\sqrt{1-x^2}} \quad (f'(x))^2 = \frac{(1-x)^2}{1-x^2} = \frac{1-x}{1+x}$$

$$1 + (f'(x))^2 = 1 + \frac{1-x}{1+x} = \frac{1 + \cancel{x} + 1 - \cancel{x}}{1+x} = \frac{2}{1+x}$$

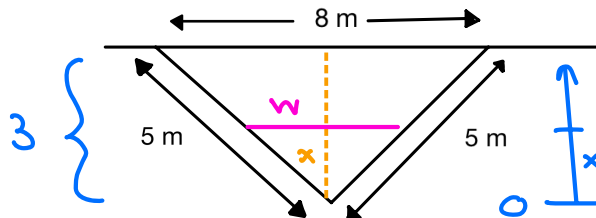
$$S = \int_0^1 \sqrt{\frac{2}{1+x}} dx = 2\sqrt{2} \sqrt{1+x} \Big|_0^1$$
$$= 2\sqrt{2} (\sqrt{2} - 1) = \boxed{4 - 2\sqrt{2}}$$

2. (20 points)

- (a) Find the y coordinate of the centroid ($\bar{y}$) of the region lying between $y = e^{-x}$ and the x -axis for $x \geq 0$.

$$\begin{aligned}
 m &= \int_0^{\infty} e^{-x} dx = -e^{-x} \Big|_0^{\infty} = 1 \\
 m_x &= \int_0^{\infty} \left(\frac{e^{-x}}{2}\right)^2 dx = \int_0^{\infty} \frac{e^{-2x}}{2} dx \\
 &= -\frac{e^{-2x}}{4} \Big|_0^{\infty} = 0 - -\frac{1}{4} = \frac{1}{4} \\
 \bar{y} &= \frac{m_x}{m} = \boxed{\frac{1}{4}}
 \end{aligned}$$

- (b) A triangle with sides 5,5 and 8 meters is submerged vertically in water ($\rho g = 9810 \text{ N/m}^3$) with the 8 meter side at the surface. Find the force on the plate.



$$h = 3 - x \qquad \frac{w}{x} = \frac{8}{3} \qquad w = \frac{8}{3} x$$

$$\begin{aligned}
 F &= \int_a^b (\rho g) h(x) w(x) dx \\
 &= \int_0^3 (9810)(3-x)\left(\frac{8}{3}x\right) dx \\
 &= 9810\left(\frac{8}{3}\right) \int_0^3 3x - x^2 dx = 26,160 \left[\frac{3x^2}{2} - \frac{x^3}{3} \right]_0^3 \\
 &= 26,160 \left[\frac{27}{2} - \frac{27}{3} \right] = \boxed{117,720 \text{ N}}
 \end{aligned}$$

3. (20 points)

(a) Solve $y' = \cos x + y^2 \cos x$; $y(0) = 1$

$$\frac{dy}{dx} = \cos x (1 + y^2) \quad \int \frac{1}{1+y^2} dy = \int \cos x dx$$

$$\text{ARCTAN } y = \sin x + C$$

$$y = \tan(\sin x + C) \quad 1 = \tan(C) \quad C = \frac{\pi}{4}$$

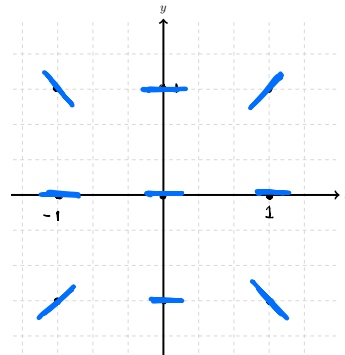
$$y = \tan\left(\sin x + \frac{\pi}{4}\right)$$

(b) For the differential equation:

$$\frac{dy}{dx} = xy$$

i. Fill in the chart below and sketch the slope field for $x = -1, 0, 1$ and $y = -1, 0, 1$

	x		
	-1	0	1
1	-1	0	1
0	0	0	0
-1	1	0	-1



ii. If $y(-1) = 1$, use Euler's method with $h = 1$ to approximate $y(1)$.

x	y	$F(x, y)$	$hF(x, y)$	$y + hF(x, y)$
-1	1	-1	-1	0
0	0	0	0	0
1	0			

$$y(1) \approx 0$$

4. (20 points)

- (a) Anthony jumps out of a plane without a parachute (we don't know why). His velocity can be modeled by the differential equation

$$\frac{dv}{dt} = -0.24(v - v_T)$$

where v_T represents the constant terminal velocity.

At time $t = 0$, Anthony's speed is 12 m/s. Assuming the terminal velocity is $v_T = 55$ m/s, determine the time at which Anthony's speed reaches 52 m/s.

$$\begin{aligned}\frac{dv}{dt} &= -0.24(v - 55) & \int \frac{1}{v - 55} dv &= \int -0.24 dt \\ e^{\ln(v - 55)} &= -0.24t + C & v &= 55 + C e^{-0.24t} \\ 12 &= 55 + C e^{-0.24(0)} & C &= -43 \\ v &= 55 - 43 e^{-0.24t} \\ 52 &= 55 - 43 e^{-0.24t} & t &= 11 \text{ sec}\end{aligned}$$

- (b) At Case, there are 462 computer science students. An outbreak of norovirus begins due to shared use of computer lab equipment. Initially, 22 students are diagnosed with the virus. Three days later, the number of cases rises to 112. Assuming the spread of norovirus follows a logistic growth model and remains confined to the computer science department, how many students will be infected after 7 days?

$$\begin{aligned}A &= 462 \\ k &= ? \\ y_0 &= 22 \\ B &= \frac{22}{22 - 462} = -\frac{1}{20} \\ y &= \frac{462}{1 + 20e^{-kt}} \\ 112 &= \frac{462}{1 + 20e^{-k(3)}} \\ k &= 0.6187 \\ y(7) &= \frac{462}{1 + 20e^{-0.6187(7)}} = 366 \text{ students}\end{aligned}$$

5. (20 points)

(a) Solve $xy' + 3y = 2x^3$ $y(1) = 1$

$$\begin{aligned}
 y' + \frac{3}{x}y &= 2x^2 & P(x) &= \frac{3}{x} & Q(x) &= 2x^2 \\
 \alpha(x) &= e^{\int \frac{3}{x} dx} = e^{3 \ln x} = x^3 \\
 y &= \frac{1}{x^3} \left[\int x^3 \cdot 2x^2 dx + C \right] = \frac{1}{x^3} \left[\int 2x^5 dx + C \right] \\
 y &= \frac{1}{x^3} \left[\frac{2x^6}{6} + C \right] = \frac{x^3}{3} + \frac{C}{x^3} \\
 1 &= \frac{1}{3} + C & C &= \frac{2}{3} & y &= \frac{x^3}{3} + \frac{2}{3x^3}
 \end{aligned}$$

(b) Sophie is making a vat of apple juice, but she wants to add more lemon zest. The vat starts with a solution of 10 gallons of apple juice and 3 ounces of lemon zest. She pours in a solution of apple juice with 5 ounces of zest per gallon at a rate of 2 gallons per minute. The tank is drained at a rate of 2 gallon per minute. How much lemon zest is in the vat after 12 minutes?

$\begin{array}{l} 2 \text{ gal/m} \\ 5 \text{ oz/g} \end{array}$

 $y = \# \text{ OF OZ OF LEMON ZEST IN TANK}$

$$\frac{dy}{dt} = R_1 - R_0 = 2 \cdot 5 - 2 \frac{y}{10} \quad y(0) = 3$$

$$\begin{aligned}
 \frac{dy}{dt} + \frac{1}{5}y &= 10 & P(t) &= \frac{1}{5} & Q(t) &= 10 \\
 \alpha(t) &= e^{\int \frac{1}{5} dt} = e^{\frac{1}{5}t} \\
 y &= \frac{1}{e^{\frac{1}{5}t}} \left[\int e^{\frac{1}{5}t} \cdot 10 dt + C \right] = e^{-\frac{1}{5}t} \left[50 e^{\frac{1}{5}t} + C \right] \\
 y &= 50 + C e^{-\frac{1}{5}t} \\
 3 &= 50 + C e^{-\frac{1}{5}(0)} & C &= -47 \\
 y &= 50 - 47 e^{-\frac{1}{5}t} \\
 y(12) &= 50 - 47 e^{-\frac{1}{5}(12)} = 45.7 \text{ oz.}
 \end{aligned}$$