SI:

Math 122 Test 2

EF:

March 5, 2024

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Name KEY

Directions:

1. No books, notes or figure skating. You may use a calculator to do routine arithmetic computations. You may *not* use your calculator to store notes or formulas. You may not share a calculator with anyone.

2. You should show your work, and explain how you arrived at your answers. A correct answer with no work shown (except on problems which are completely trivial) will receive no credit. If you are not sure whether you have written enough, please ask.

3. You may not make more than one attempt at a problem. If you make several attempts, you must indicate which one you want counted, or you will be penalized.

4. You may leave as soon as you are finished, but once you leave the exam, you may not make any changes to your exam.

5. This test has 10 problems.

1. Consider the probability density function $p(x) = C\sqrt{x-4}$ on the interval $4 \le x \le 13$.

   (a) For what value of C is $p(x)$ a valid probability density function?

   $$\int_4^{13} C\sqrt{x-4}\ dx = C\frac{2(x-4)^{\frac{3}{2}}}{3}\Big|_4^{13} = \frac{2C}{3}\left[27-0\right]$$

   $$= 18C = 1 \qquad C = \frac{1}{18}$$

   (b) Find the mean μ.

   $$N = \int_4^{13} \frac{1}{18} x\sqrt{x-4}\ dx \qquad \begin{array}{l} U = x-4 \\ du = dx \end{array} \qquad x = U+4$$

   $$= \int \frac{1}{18}(U+4)\sqrt{U}\ dU = \frac{1}{18}\int U^{\frac{3}{2}} + 4U^{\frac{1}{2}}\ dU = \frac{1}{18}\left[\frac{2U^{\frac{5}{2}}}{5} + 4\frac{2U^{\frac{3}{2}}}{3}\right]$$

   $$= \frac{1}{18}\left[\frac{2}{5}(x-4)^{\frac{5}{2}} + \frac{8}{3}(x-4)^{\frac{3}{2}}\right]_4^{13} = \frac{1}{18}\left[\frac{486}{5} + 72\right] = \frac{47}{5}$$

2. (10 points) Find the value of D so the arc length of the curve

   $$f(x) = \frac{2}{3}(x-4)^{3/2}$$

   for $4 \le x \le D$ is $\frac{52}{3}$.

   $$f'(x) = (x-4)^{\frac{1}{2}} \qquad (f'(x))^2 = x-4 \qquad 1 + (f'(x))^2 = 1 + x - 4$$
   $$= x-3$$

   $$S = \int_4^D \sqrt{x-3}\ dx = \frac{2(x-3)^{\frac{3}{2}}}{3}\Big|_4^D = \frac{2}{3}\left[(D-3)^{\frac{3}{2}} - 1\right]$$

   $$\frac{2}{3}\left[(D-3)^{\frac{3}{2}} - 1\right] = \frac{52}{3} \qquad (D-3)^{\frac{3}{2}} = 27$$

   $$D-3 = 9 \qquad D = 12$$

3. (10 points) Find the value of B so the x-coordinate of the centroid of the region bounded by $y = x^{2/3}$, $y = 0$ and $x = B$ ($B > 0$) is 5.

$$M = \int_0^B x^{\frac{2}{3}} \, dx = \frac{3x^{\frac{5}{3}}}{5} \Big|_0^B = \frac{3}{5} B^{\frac{5}{3}}$$

$$M_y = \int_0^B x \cdot x^{\frac{2}{3}} \, dx = \int_0^B x^{\frac{5}{3}} \, dx = \frac{3x^{\frac{8}{3}}}{8} \Big|_0^B = \frac{3}{8} B^{\frac{8}{3}}$$

$$\bar{x} = \frac{M_y}{M} = \frac{\frac{3}{8} B^{\frac{8}{3}}}{\frac{3}{5} B^{\frac{5}{3}}} = \frac{5}{8} B = 5 \qquad B = 8$$

4. (10 points) A thin plate in the shape of an isosceles triangle with base 3 meters and height 5 meters is submerged so the at the top of the triangle is 4 meters below the surface of the water ($\rho g = 9810$ N/m³). Find the force on the plate.

$$F = \int_a^b (\rho g) h(x) w(x) \, dx$$

4 meters

5 meters

3 meters

$$h(x) = 4 + x$$

$$\frac{w}{x} = \frac{3}{5} \qquad w = \frac{3x}{5}$$
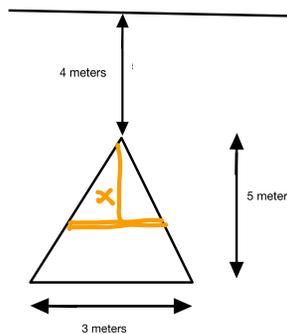
$$F = \int_0^5 (9810)(4+x) \frac{3x}{5} \, dx = \frac{9810 \cdot 3}{5} \int_0^5 4x + x^2 \, dx$$

$$= \frac{9810 \cdot 3}{5} \left[2x^2 + \frac{x^3}{3} \right]_0^5 = \frac{9810 \cdot 3}{5} \left[50 + \frac{125}{3} \right]$$

$$= 539{,}550 \text{ N}$$

5. (10 points) $\frac{dy}{dx} = (x-1)(y-2)$ $y(0) = 3$

$$\int \frac{1}{y-2} dy = \int (x-1) dx$$

$$3 = 2 + Ce^0$$

$$C = 1$$

$$\ln|y-2| = \frac{x^2}{2} - x + C$$

$$y = 2 + e^{\frac{x^2}{2} - x}$$

$$e \qquad e$$

$$y - 2 = Ce^{\frac{x^2}{2} - x}$$

$$y = 2 + Ce^{\frac{x^2}{2} - x}$$

6. (10 points) A big (I mean really big!!!) figure skating competition is happening in 2 hours. We need to get the water frozen (0° C). If the initial temperature of the water is 10° C, and after 1 hour it is 3° C, what must the room temperature be s the water is frozen after 2 hours?

$$y(0) = 10 \qquad y = T_0 + Ce^{-kt}$$
$$y(1) = 3$$
$$y(0) = 0 \qquad 10 = T_0 + C \qquad 3 = T_0 + Ce^{-k} \qquad 0 = T_0 + Ce^{-2k}$$

$$\frac{10 - T_0 = C}{3 - T_0 = Ce^{-k}} = e^k \qquad \frac{3 - T_0 = Ce^{-k}}{-T_0 = Ce^{-2k}} = e^k$$
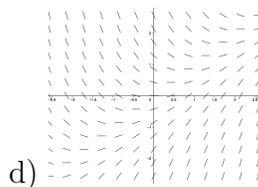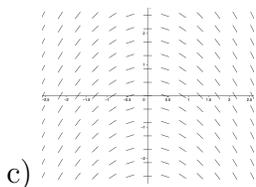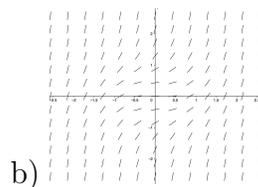
$$\frac{3 - T_0}{-T_0} = \frac{10 - T_0}{3 - T_0} \implies 9 - 6T_0 + T_0^2 = -10T_0 + T_0^2$$
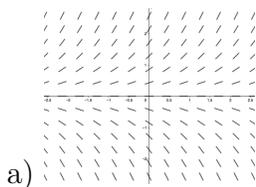
$$4T_0 = -9$$

$$T_0 = -\frac{9}{4} = -2.25^{\circ}C$$

7. (10 points) Match the differential equations with its slope field:

a)

b)

c)

d)

I) $y' = y$ ___a___

. II) $y' = -x$ ___c___

III) $y' = x - y$ ___d___

IV) $y' = x^2 + y^2$ ___b___

8. (10 points) There is a small town (with nothing else to do) with a growing interest in figure skating. The number of individuals participating in figure skating lessons is modeled by a logistic growth equation. The carrying capacity is 300. The initial number of figure skaters is 30, and after 10 years, there are 60 figure skaters. When will there be 100 figure skaters?

$K =$

$A = 300$

$Y_o = 30$

$B = \dfrac{30}{30-300} = -\dfrac{1}{9}$

$y = \dfrac{300}{1+9\,e^{-Kt}}$

$60 = \dfrac{300}{1+9\,e^{-K(10)}}$

$k = .0811$

$100 = \dfrac{300}{1+9\,e^{-.0811\,t}}$

$t = 18.54$ YEARS

9. (10 points) $\frac{dy}{dx} - 2xy = x$ $y(0) = 2$ $P(x) = -2x$ $Q(x) = x$

$$\alpha(x) = e^{\int -2x \, dx} = e^{-x^2}$$

$$y = \frac{1}{e^{-x^2}} \left[\int e^{-x^2} \cdot x \, dx + c \right]$$

$$y = e^{x^2} \left[-\frac{e^{-x^2}}{2} + c \right]$$

$$u = -x^2$$
$$du = -2x \, dx$$

$$y = -\frac{1}{2} + ce^{x^2} \qquad 2 = -\frac{1}{2} + c$$

$$C = \frac{5}{2}$$

$$y = -\frac{1}{2} + \frac{5}{2} e^{x^2}$$

10. (10 points) A Zamboni machine is in the process of cleaning an ice rink, which initially contains 3,000 gallons of ice. The Zamboni replaces 150 gallons of ice per hour, with a continuous exchange of 150 gallons in and 150 gallons out. Let $y = y(t)$ represent the quantity of new ice in the rink, where $y(0) = 0$ is the initial amount of new ice.

(a) Write a differential equation for how much new ice is in the rink.

$$\frac{dy}{dt} = 150 - 150 \frac{y}{3000} = 150 - \frac{y}{20}$$

(b) Solve the differential equation from part (a).

$$P(t) = \frac{1}{20} \qquad Q(t) = 150 \qquad \alpha(t) = e^{\int \frac{1}{20} dt} = e^{\frac{1}{20}t}$$

$$y = \frac{1}{e^{\frac{1}{20}t}} \left[\int 150 e^{\frac{1}{20}t} \, dt + c \right] = e^{-\frac{1}{20}t} \left[3000 e^{\frac{1}{20}t} + c \right]$$

$$y = 3000 + ce^{-\frac{1}{20}t} \qquad 0 = 3000 + c \qquad c = -3000$$

$$y = 3000 - 3000 e^{-\frac{1}{20}t}$$

(c) How long will it take for the new ice to account for 90% of the ice in the rink?

$$2700 = 3000 - 3000 e^{-\frac{1}{20}t}$$

$$t = 46 \text{ Hours}$$

FORMULA PAGE

$$\sin^2\theta + \cos^2\theta = 1$$

$$\tan^2\theta + 1 = \sec^2\theta$$

$$1 + \cot^2\theta = \csc^2\theta$$

$$\sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$$

$$\sin(\alpha - \beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$$

$$\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$

$$\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$$

$$\tan(\alpha + \beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}$$

$$\sin^2 x = \frac{1 - \cos 2x}{2}$$

$$\cos^2 x = \frac{1 + \cos 2x}{2}$$

$$\sin 2x = 2\sin x\cos x$$

$$(\sin x)' = \cos x$$

$$(\cos x)' = -\sin x$$

$$(\tan x)' = \sec^2 x$$

$$(\sec x)' = \sec x\tan x$$

$$(\csc x)' = -\csc x\cot x$$

$$(\cot x)' = -\csc^2 x$$

$$(e^x)' = e^x$$

$$(\ln x)' = \frac{1}{x}$$

$$(\arcsin x)' = \frac{1}{\sqrt{1 - x^2}}$$

$$(\arctan x)' = \frac{1}{1 + x^2}$$

$$(\operatorname{arcsec} x)' = \frac{1}{|x|\sqrt{x^2 - 1}}$$

$$\sinh x = \frac{e^x - e^{-x}}{2}$$

$$\cosh x = \frac{e^x + e^{-x}}{2}$$

$$\cosh^2 x - \sinh^2 x = 1$$

$$(\sinh x)' = \cosh x$$

$$(\cosh x)' = \sinh x$$

$$(\operatorname{arcsinh} x)' = \frac{1}{\sqrt{x^2 + 1}}$$

$$(\operatorname{arccosh} x)' = \frac{1}{\sqrt{x^2 - 1}}$$

$$(\operatorname{arctanh} x)' = \frac{1}{1 - x^2}$$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

$$f(c) = \frac{1}{b - a}\int_a^b f(x)dx$$

$$\int \sec x\, dx = \ln|\sec x + \tan x| + C$$

$$\int \sec^3 x\, dx = \frac{1}{2}\left[\sec x\tan x + \ln|\sec x + \tan x|\right] + C$$

$$\int \csc x\, dx = \ln|\csc x - \cot x| + C$$

$$y' = ky\left(1 - \frac{y}{A}\right)$$

$$y = \frac{A}{1 - \dfrac{e^{-kt}}{B}} \qquad B = \frac{y_0}{y_0 - A}$$

$$1 + 1 = 2$$