What follows is Vladimir Voevodsky's snapshot of his Fields Medal work on *motivic homotopy*, plus a little philosophy and "from my point of view the main fun of doing mathematics" Voevodsky (2002). Voevodsky uses extremely simple leading ideas to extend topological methods of *homotopy* into number theory and abstract algebra. The talk takes the simple ideas right up to the edge of some extremely refined applications. The talk is on-line as streaming video at claymath.org/video/.

In a nutshell: classical homotopy studies a topological space M by looking at maps to M from the unit interval on the real line:

$$[0,1] \;=\; \{x \in \mathbb{R} \mid 0 \leq x \leq 1\;\}$$

as well as maps to M from the unit square $[0,1]\times[0,1]$, the unit cube $[0,1]^3$, and so on. Algebraic geometry cannot copy these naively, even over the real numbers, since of course the interval $[0,1]$ on the real line is not defined by a polynomial equation. The only polynomial equal to 0 all over $[0,1]$ is the constant polynomial 0, thus equals 0 over the whole line. The problem is much worse when you look at algebraic geometry over other fields than the real or complex numbers. Voevodsky shows how category theory allows abstract definitions of the basic apparatus of homotopy, where the category of topological spaces is replaced by any category equipped with objects that share certain purely categorical properties with the unit interval, unit square, and so on. For example, the line in algebraic geometry (over any field) does not have "endpoints" in the usual way but it does come with two naturally distinguished points: 0 and 1. Voevodsky's approach will make this work "like" the interval.

He makes any category with suitable objects define a homotopy theory. For each case it remains that "how good this definition will be is a big question," and in fact "in most cases it is not going to work very well, but it will work somehow" (Voevodsky, 2002, minutes 23, 26). It has worked very well in some cases. For relatively accessible accounts of more see Voevodsky (1998) and Soulé (2003).

An Intuitive Introduction to Motivic Homotopy Theory

Vladimir Voevodsky

I was given by Arthur sort of an impossible task. I was requested to explain motivic homotopy theory so that it will be understood by non-mathematicians. So what came out of that you will see now but I do not guarantee that it will be understandable either by mathematicians or non-mathematicians. (minute 2)

At first there were spaces. Mathematicians studied spaces and starting with simple examples of spaces. Some are shown in figure 1. First is one of the most basic spaces, a point. There is another which is more basic but it is hard to draw—because it is empty. Second is another standard space which I will talk about, the unit interval on the real line, the line between 0 and 1. The third example in the figure is a surface, a space of dimension 2, which is supposed to look like some baked good with two holes in it. The last example is more mathematical. It is the space $\mathrm{GL}_n(\mathbb{C})$ of all $n \times n$ complex matrices of non-zero c determinant. That is a (minute 3) sophisticated space which, depending on how you count, has dimension either n^2 or $2n^2$.

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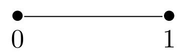
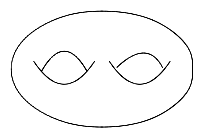
(2) 

(3) 

(4) $\mathrm{GL}_n(\mathbb{C})$
the space of $n \times n$ complex number matrices with non-zero determinant

FIGURE 1. Some Spaces

There are many different axiomatizations of the notion of a space in mathematics. (minute 4) For instance there are topological spaces, there are metric spaces. In the end it does not matter as long as one only considers non-pathological examples it all comes to the same thing. So I will assume there is some intuitive understanding of what space is.

Any space, if you formalize it by really writing down a definition will be a tremendously infinite object so it is very difficult to work with. So to distinguish spaces mathematically we use what we call *invariants* which are something more understandable and typically more finite than the space itself. There are two invariants (minute 5) I will dare to call the most fundamental. One is the *dimension* of the space. The point has dimension 0, and the interval has dimension 1.

$$\dim(\bullet) = 0$$

$$\dim(\bullet\!\!-\!\!-\!\!-\!\!\bullet) = 1$$

FIGURE 2. Dimension

Another invariant is the number of pieces the space consists of. The point consists of one piece. You cannot divide it into disconnected pieces. And here is sort of a deformed circle with a deformed interval sitting inside it, so there are two pieces, two disjoint pieces, namely the circle and the interval. Typically when one thinks of a space one thinks of a space with only one piece. But that is just a custom.

(minute 6) What is homotopy theory about, what does it study? First let us say π_0 of a space is the set of its pieces. So if a space has two pieces then π_0 is a set of two elements, namely the pieces. So it is typically more finite than the space itself. Homotopy theory studies some fancy versions of this invariant which are called

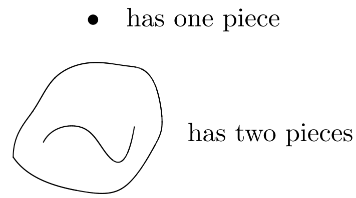


FIGURE 3. The number of "pieces"

higher homotopy sets π_n. I will try to explain how one gets from π_0 to these π_n where π_n is a set for every non-negative integer n.

It can be done in different ways. I choose to base my explanation on the notion of *continuous map* which will also play a fundamental role later on. What is a (minute 7) continuous map? I have spaces M and N and a map $f : M \to N$ is any rule which assigns to each point of M a point of N. And the idea is that f is continuous if whenever points are close in M they do not get too far apart in N. The easiest way to understand it is to look at examples.

So let $C(M, N)$ denote the set of all continuous maps from M to N and see some examples. Look at continuous maps $C(pt, N)$ from a point to any space N. A map from a point to a space is just determined by where this unique point goes. So a (minute 8) map from a point to a space N is the same as a point of N. So the set of maps $C(pt, N)$ from a point to N is just the set of points of N.

$$C(pt, N) \;=\; \text{set of points of } N$$

Another example is the set of maps $C(N, pt)$ from any space N to a point. There is only one place for all the points of N to go, namely the sole point of the target space, so there is always only one map from any space to the point.

$$C(N, pt) \text{ always has just one element}$$

Here is a fancier example. Let us look at maps from the unit interval to any space N where you may think of N as a surface outlined in Figure 4.

$$C([0, 1], N)$$

Any map $\gamma : [0, 1] \to N$ will have some beginning point $\gamma(0)$ where the point 0 goes, and some end point $\gamma(1)$ where the point 1 goes, and there will be all the points along this path where the numbers between 0 and 1 go. (minute 9)

A continuous map $\gamma : [0, 1] \to N$

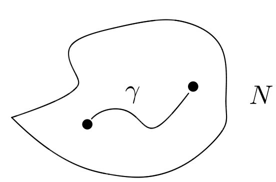


FIGURE 4. An example of a path γ in $C([0, 1], N)$

The fact that it is a continuous map means that this path has no breaks in it. That is what it means intuitively. So the set $C([0,1],N)$ of all continuous maps from the interval to any space N is just the set of paths in N.

Let us go back to the notion of the number of pieces of a space. How can we say formally when two points lie in the same piece? If they lie in the same piece then they can be connected by a path. If they lie in different pieces there is no way to find a path without breaks which connects one with the other. So to define formally what it means to say two points lie in the same piece one can use this notion of a
(minute 10) path, of a continuous map from the interval to the space. Let me repeat that: The only thing which one needs to know to define the set π_0 of N, the set of pieces of N, is to know the points and the paths between these points.

Now I move to the next homotopy set, π_1, which is a little bit fancier and harder to explain. As I said, π_0 is the first of an infinite series of homotopy sets. For π_0
(minute 11) one takes points and identifies any two that can be connected by a path. For π_1 suppose I fix two points, p and q, which lie in the same piece. Then I consider

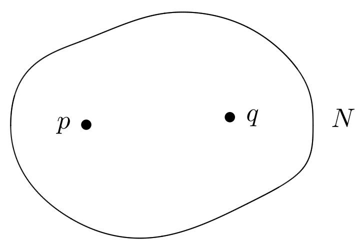


FIGURE 5. Two points in one piece of a space N

paths that connect p to q. Let us look at all the paths. So we get some space of

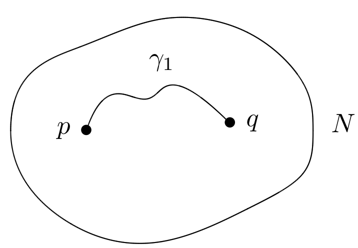


FIGURE 6. A path γ_1 connecting p to q

paths whatever that means. Let us try to understand how many pieces this space of paths has, and call this set of pieces the π_1 of the space relative to p and q.

In practice what one has to do is consider continuous maps from the square to the space, and let us say I request maps which contract the left hand side completely to point p and the right hand side completely to point q. Then the image of the top side will be a path from p to q, call that γ_1. And the image of the bottom side
(minute 12) will also be a path from p to q, call that γ_2. And as I move from the top side to the bottom side continuously I can think of that as being a "path between the paths." It is a continuous deformation between the top path and the bottom path.

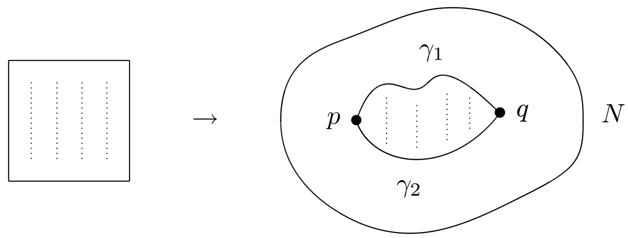


FIGURE 7. A map of the square into N which contracts the two vertical sides to the points p and q, takes the top side of the square to the path γ_1, and takes the bottom side to the path γ_2

Now I can consider the set of pieces π_1 of the space of paths, where two paths are declared to belong to the same piece if they can be connected by such a map from the square. This procedure can be repeated and I can use cubes to connect maps from squares, and so on. One gets a sequence of sets π_n which are called the homotopy sets of the space.

There is one nice example which I want to give. If we try to define these homotopy (minute 13) sets for the space $\mathrm{GL}_n(\mathbb{C})$ of complex matrices with non-zero determinant it is of course a very complicated problem to compute such a weird thing for such a weird space. However it turns out it can be done at least for very very big matrices, that is matrices of size $n \times n$ where n is very large, and the answer turns out to be wonderfully simple. The first homotopy set is the point, any two matrices can be connected by a path. There is only one piece in the whole space of matrices. If I (minute 14) consider paths then it runs out that the pieces in the space of paths are number by the integers, so π_1 can be identified with the set of integers. It turns out π_2 is again just one point and π_3 is again the set of integers, and so on for a very long time. How long depends on the number n but basically if n is infinity then it goes on forever. That is the Bott periodicity theorem. That was one of the very impressive (minute 15) achievements of mathematics in, maybe the early 1960s or late 1950s. That was one of the first really big computations of homotopy theory.

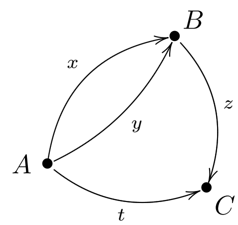
$$
\begin{aligned}
\pi_0(\mathrm{GL}_n(\mathbb{C}), 1, 1) &= \text{a point} \\
\pi_1(\mathrm{GL}_n(\mathbb{C}), 1, 1) &= \text{set of integers} \\
\pi_2(\mathrm{GL}_n(\mathbb{C}), 1, 1) &= \text{a point} \\
\pi_3(\mathrm{GL}_n(\mathbb{C}), 1, 1) &= \text{set of integers} \\
&\vdots
\end{aligned}
$$

FIGURE 8. Homotopy sets of $\mathrm{GL}_n(\mathbb{C})$

That is what I wanted to say about homotopy theory of spaces. It studies spaces. It associates to them these homotopy sets which come from studying paths of paths and so on. And here is an example of a computation. These are actually things that can be calculated in some nontrivial cases. Now I want to jump to do something similar in the algebraic setting.

There is only one known way to do it. I don't know how much I will have time
(minute 16) to say today. But there is one topic which I want to talk about very quietly and without hurry. That is something which I never saw even *mentioned* in any popular book about mathematics. I saw algebraic equations mentioned, I saw probably even homotopy groups mentioned—especially in elementary books on topology. But I think that at the heart of 20th century mathematics lies one particular notion and that is the notion of a category. So let me try to explain what it is about. It is a very formal gadget and I will illustrate it by an example.

(minute 17) Here are two examples of categories. First I have a picture of one category.



A, B, C — objects

x, y, z, t — morphisms

$xz = t$ — composition rules

$yz = t$

FIGURE 9. First example of a category

There are three objects A, B, C and there are arrows between those objects which are called morphisms and then there is the rule which says that if I have one arrow and it is followed by another arrow, then there is a third arrow. Let's say I have an arrow from A to B and an arrow from B to C then there is a rule to get from such a pair an arrow from A to C. That is called a *composition* rule. But let's forget this third part for a second. It is not necessarily important for intuitive understanding. The main thing that exists in categories is that there are objects and there are morphisms. There is a formal set, in this case $\{A, B, C\}$, which is called the set of
(minute 18) objects. For every pair of objects there is another formal set, in this case it is

- $\{x, y\}$ between A and B
- $\{z\}$ between B and C
- $\{t\}$ between A and C

which are called morphisms between these objects.

The first example is an example of a combinatorial, finite category. Here is a second example. The objects are spaces, all the spaces that exist. Morphisms between two spaces are continuous maps between those spaces. And of course continuous maps can be composed. So that gives an example of a category of a very different kind.

Spaces are objects
Continuous maps are morphisms
Obvious composition.

FIGURE 10. Second example of a category

What was somewhat amazing for me, personally, an amazing discovery, was that for almost all—let us say for very many—classes of mathematical objects there are natural categories which correspond to them. So for most classes of mathematical objects we can construct a category whose objects will be these objects, morphisms will be some natural way of connecting one object to another, and there will be some kind of composition. (minute 19)

Here is a more formal way to say what a category $\mathcal{X}$ is. There is a set of objects and for any pair of objects there is a set of morphism between those objects and then there is a composition rule.

- a set of objects of $\mathcal{X}$
- for any pair of objects N and M a set of morphisms from N to M

$$\mathrm{Mor}(N, M)$$

- a rule giving for each morphism $N \xrightarrow{f} M$ and morphism $M \xrightarrow{g} K$ the *composite* $N \xrightarrow{fg} K$

FIGURE 11. To specify a category $\mathcal{X}$

What I hope to do now is show how this works, and why it is interesting. Let us for instance consider the category of topological spaces. And let us choose a space N. What can we say about this space looking at it exclusively as being an object of the category of spaces? So I do not want to look inside this space, I know nothing about its inside. I only know continuous maps relating this space to all other spaces. Instead of looking at the structure of the object you look at its sort of "social relationships!" (minute 20)

As a very first example, can we figure out when a space is a point? There is a nice answer: A space N is a point if and only if for every other space there is a single morphism to N. That characterizes a point exclusively by morphisms to and from it. (minute 21)

$$N = pt \quad \Leftrightarrow \quad \text{for each } M,\ \mathrm{Mor}(M, N) \text{ has exactly one element}$$

FIGURE 12. When is N a point?

Let us ask a more complicated question. When is a space N connected? When does it consist of only one piece? Or more generally, can we somehow express let us say the first homotopy set, the set of pieces, purely in categorical terms? It turns out yes we can. So let us try to play this game.

First of all we know the set of points of the space N. It is the set of morphisms from the point to this space. We have already characterized the point in an internal way. Now let us consider the unit interval $[0, 1]$. The set of morphisms from the unit interval to the space is the set of paths. I need to know the beginning and the end of a path. For that I need the notion of composition of morphisms. If I have a path, I have two distinguished points 0 and 1 on the unit interval, which are morphisms from the point to the unit interval. If I look at the composition of my path with one of these points I get a point of the space which is the beginning or the end of the path. (minute 22)

- $\mathrm{Mor}(pt, N)$ — the set of points of N

  Consider two points $pt \xrightarrow{a} [0,1]$ and $pt \xrightarrow{b} [0,1]$.
- $[0,1]$ has distinguished points

  $$pt \xrightarrow{0} [0,1] \text{ and } pt \xrightarrow{1} [0,1]$$
- $\mathrm{Mor}([0,1], N)$ — the set of paths of N
- for $[0,1] \xrightarrow{\gamma} N$

  $\gamma(0)$ — beginning point
  $\gamma(1)$ — end point.
- Points a, b lie in the same "piece" of N if there is a path γ with $a = \gamma(0)$ and $b = \gamma(1)$.

FIGURE 13. How to compute $\pi_0(N)$ in categorical terms

So using purely the structure of the category I know the set of points, I know (minute 23) the set of paths, and I know how to assign the beginning and end of the path. So now I can say what the set of pieces is because I simply have to identify any two points which can be connected by a path, in other words for which there is a path with them as beginning and end. And that gives me the set π_0.

The only thing here which is not internal to the categorical structure is the unit interval itself. We do not know how to characterize it from the inside. But as soon as I have a category and some object in it which I want to call the *unit interval* with two maps 0 and 1 called beginning and end, I can define some kind of π_0. How good this definition will be is a big question.

(minute 24) Similarly, if I want to compute the next homotopy set (here I will not go into much detail) I need to know the object *pt* called *point*, the object $[1,0]$ called *unit interval*, and I need some distinguished object which I might write I^2 which is called the *square* and some finite number of morphisms connecting these objects. For the square I need four maps $[0,1] \to I^2$ which I call the sides of the square. As soon as I have such a purely combinatorial structure inside my category I can consider morphisms with values in any object and get something which I will call π_0 and π_1. I can continue that and model formally in any category each of the π_n. But basically for what I am talking about it is enough to understand π_0, and the rest (minute 25) is not too elementary.

So one can formalize the situation where one has point, interval, square, cube, cube of the next dimension, and so on. All of those things in the usual topological world map to each other. There are two distinguished maps from the point to the interval – the beginning and the end. There are four maps from the interval to the square corresponding to the four sides of the square. There are six maps from the square to the cube corresponding to the six sides of the cubes. And so on. These satisfy some relations.

So I can formally say let us take any category whatsoever, let us take a series of objects and morphisms between them satisfying appropriate conditions. Let us call such a gadget a *cubical object*. As soon as I have a cubical object in any category I can play the game which I explained for π_0 and π_1 and define some kind (minute 26) of homotopy sets for any object in this category. Of course it needs to be said that in most cases this is not going to work very well. But it will work somehow.

Given any category and a cubical object in it we can define analogs of the homotopy sets π_n for all objects of the category.

FIGURE 14. Observation

Of course the π_n are just an example. In principle there are many more structures, let us say in topological spaces, which are needed for homotopy theory. One (minute 27) can translate most of these structures into categorical terms, as I tried to show in this example of cubical spaces. As soon as they are translated into these formal categorical terms one can try to take some very different category and apply it there and see what happens.

Motivic homotopy theory happens when one applies this procedure to the following category of systems of algebraic equations. This is not exactly the standard approach to algebraic geometry but it is the most elementary one I could think of.

Let us consider the following objects which we call systems of algebraic equations. By definition a system of algebraic equations is the following gadget: first, a list of letters which I call *variables* of the system, and second a list of polynomials in the variables which I call *equations* of the system. So here is an example. There (minute 28) is one variable x, and one equation $x^2 + 1$. Here is another example. There are n variables $x_1 \dots x_n$ and there are no equations whatsoever. This is a very important system denoted by $\mathbb{A}^n$.

A *system of algebraic equations* consists of:

- A list of letters called *variables*
  $x_1, \dots, x_n$
- A list of polynomials in these variables
  $f_1, \dots, f_m$ called *equations*.
- One example: $(x; x^2 + 1)$
- Another example is $(x_1, \dots, x_n;)$ with no equations. This is called $\mathbb{A}^n$.
- A more sophisticated example
  $(x_{11}, x_{12}, \dots, x_{nn}, t; \mathrm{Det}(x_{ij}) \cdot t - 1)$.
  This system is called GL_n.

FIGURE 15. Systems of algebraic equations

Here is a more sophisticated example. It has n^2 variables x_{ij} thought of as entries in a matrix, plus one additional variable t, and one equation

$$\mathrm{Det}(x_{ij}) \cdot t - 1$$

which in effect says the determinant cannot be 0, since multiplying it by t gives 1. But actually it does not *say* anything, it is just my polynomial. This system of equations is denoted GL_n. (minute 29)

Now I want to define the category of systems of algebraic equations. As objects I sort of want to take all systems of algebraic equations but that is a bit too many. So a typical approach is first to choose what kind of coefficients we will consider. So let us take some system of coefficients, say integers or real numbers or complex numbers, and consider all systems with coefficients in these numbers. That will be (minute 30) the class of objects of our category. Now I need morphisms from one system to another. I want a morphism to be a change of variables. A morphism from one system in variables x to another system in variables y is going to be some way of expressing variables y in terms of variables x in such a way that the equations sort of are satisfied on both sides. There is an obvious way to compose these things. I will not explain this in detail. It was not I who invented this. I do not know who invented it. But I read it in a wonderful textbook by Manin on algebraic geometry (minute 31) published in the Soviet Union at the end of the 1960s.

Here is an example. Let us consider $\mathbb{A}^0$. Remember $\mathbb{A}^0$ is the system of no equations and no variables. So a morphism from $\mathbb{A}^0$ to another system would mean finding an expression of the variables of that second system, in term of in terms of no variables whatsoever from $\mathbb{A}^0$. It means I have to find constants of the second system which satisfy the equations of the second system. Such a morphism is exactly a solution of the second system of equations. similarly one can verify that (minute 32) every system of equations has exactly one morphism to $\mathbb{A}^0$.

For any algebraic system of equations X:

- $\mathrm{Mor}(X, \mathbb{A}^0)$ has exactly one element.
- $\mathrm{Mor}(\mathbb{A}^0, X)$ = the set of solutions to X.

Figure 16. Morphisms to and from $\mathbb{A}^0$

It turns out that these objects $\mathbb{A}^n$, which are the systems of algebraic equations with no equations, form a cubical object in a very naive sense. One can easily build maps between them, in the category of systems of algebraic equations, corresponding to all the maps between usual cubes in spaces.

I can use these cubical objects to define some kind of homotopy sets for any system of algebraic equations. In general the results will not be quite correct. That is, the results will not correspond to anything important. But there is an important example. If I apply it to the system GL_n I get very very reasonable things. I get (minute 33) sets which are called algebraic K-groups of the system of coefficients.

$$\pi_m(\mathrm{GL}_N) \; = \; \mathrm{K}^{alg}_{m+1}(k)$$

These groups have been defined in many difficult ways over the last maybe 40 years and they are still the source of a great many interesting mathematical problems and insights. The result really depends on k. So for the integers it will have to do with number theory and arithmetic. For the complex numbers or real numbers there are connections with special functions such as polylogarithms and hyperbolic geometry and I don't know what else. So these are very very interesting groups and (minute 34) give an example of how such a game can lead to objects of independent interest.

Now let me have the last slide and get a little philosophical. So that is how I see the whole picture.

We start with geometry, the category of topological spaces. We invent something about this geometrical world using our basically visual intuition. The notion of pieces comes exclusively from visual intuition. We somehow abstract it and re-write it in terms of category theory which provides this connecting language. And then we apply in a new situation, in this case in the situation of algebraic equations which

GEOMETRY

A construction from homotopy theory, e.g. π_n

↓

Categorical formulation e.g. through cubical objects

↓

Specialization to the category of algebraic equations

ALGEBRA

FIGURE 17. The whole picture

is purely algebraic. So what we get is some fantastic way to translate geometric intuition into results about algebraic objects. And that is from my point of view the main fun of doing mathematics. Thank you.

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