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En hommage à Alexandre Grothendieck

FIBERED CATEGORIES AND THE FOUNDATIONS OF NAIVE CATEGORY THEORY

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§0. Introduction. Any attempt to give "foundations", for category theory or any domain in mathematics, could have two objectives, of course related.

(0.1) *Noncontradiction*: Namely, to provide a formal frame rich enough so that all the actual activity in the domain can be carried out within this frame, and consistent, or at least relatively consistent with a well-established and "safe" theory, e.g. Zermelo-Frankel (ZF).

(0.2) *Adequacy*, in the following, nontechnical sense:

(i) The basic notions must be simple enough to make transparent the syntactic structures involved.

(ii) The translation between the formal language and the usual language must be, or very quickly become, obvious. This implies in particular that the terminology and notations in the formal system should be identical, or very similar, to the current ones. Although this may seem minor, it is in fact very important.

(iii) "Foundations" can only be "foundations of a given domain at a given moment", therefore the frame should be easily adaptable to extensions or generalizations of the domain, and, even better, in view of (i), it should suggest how to find meaningful generalizations.

(iv) Sometimes (ii) and (iii) can be incompatible because the current notations are not adapted to a more general situation. A compromise is then necessary. Usually when the tradition is very strong (ii) is predominant, but this causes some incoherence for the notations in the more general case (e.g. the notation $f(x)$ for the value of a function f at x obliges one, in category theory, to denote the composition of arrows $(f, g) \to g \circ f$, and all attempts to change this notation have, so far, failed).

(0.3) Although it seems to have been the main preoccupation of the logicians who tried to give foundations for category theory, I am only mildly interested in mere consistency, for the following reasons:

(i) Categoricians have, in their everyday work, a clear view of what could lead to contradiction, and know how to build ad hoc safeguards.

(ii) If a formal system fails to satisfy too many of the adequacy requirements, it will be totally useless; and worse, the inadequacy will probably reflect too superficial an analysis of the real activity of categoricians.

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(iii) If adequacy is achieved, in a satisfactory manner, consistency should be a by-product.

(0.4) The requirements of (0.2) in a sense force the method by which they can be achieved. Namely:

(i) A careful investigation, of a "linguistic" type, both semantic (what does category theory talk about?) and syntactic (how does it talk?). This investigation should cover a sufficiently representative part of the corpus of category theory, and a good knowledge of category theory is necessary to find what is representative.

(ii) The results of such an investigation have to be accepted as they come out, however strange they, or their consequences, may seem. (And there will indeed be some very strange statements in this paper, which will become quite natural in the adequate setting).

(iii) The formal framework has to be constructed from these results, and not forced to fit into any a priori determined, well-known structure.

It will be the case that our approach fits into such a structure, namely the "fibered categories" introduced in [Gr] and developed in [Gi]. This has to be viewed as an unexpected gift, because its results are immediately available, but also as a handicap, since its notations are fixed.

(0.5) We will call *naive category theory* (as opposed to the axiomatic setting we are trying to define) all the domain covered in actual work about categories (it is all but naive in the usual sense).

The main feature, that we want to analyse and clarify, is that when we talk about categories, we use sets as a major ingredient, although categories are not sets. (Even people who work with universes do not believe that categories are sets of the same kind as ordinary sets.)

This distinction forces us to introduce quotation marks, which we shall use in the following way: we shall call "category", "functor", "set", etc.... whatever abstract structure or notion which will turn out to represent in our setting a category, a functor, a set, etc. from naive category theory.

(0.6) In order to avoid misunderstanding, I have to emphasize a few points. This *is not* a mathematical paper. Thus, although mathematics is present everywhere, in the choice of examples, in their analysis, in references to set theory, topos theory, etc...., there is not a single proof and very few precise definitions. So, what is this paper about?

(0.7) We will show, on the basis of a precise "linguistic" analysis, that:

(i) In naive category theory many assumptions are implicitly made, in a quite fundamental, but obscure, manner, which must be clarified before any foundations is sought.

(ii) Such an analysis permits us to predict that any axiomatization of naive category theory should present some essential features (e.g. definability) and also how it should behave in many important questions (e.g. (8.11)).

(iii) A clarification of type (i) of two basic notions of naive category theory (families of objects or maps of a category, and equality of objects in a category) will force any axiomatization to contain, as major features, either primitive notions or derived notions, both fibrations and split fibrations.

(iv) Apart from the axioms of fibrations, some other axioms will be needed, whose meaning and function can be understood clearly, before they are precisely stated, and even if we do not know yet how to state them (such an understanding will be a precious guide to finding what they are).

(v) This analysis also shows that some category playing the role of sets will be essential, and that we may have to make strong assumptions about it, as a category, but nothing forces us to take a model of ZF, nor, more importantly, to state these assumptions in the language of set theory, thus leaving the possibility of choosing, say, an elementary topos.

(0.8) Although we think that (0.7) fully justifies our analytic approach, we are quite aware of a major gap in this paper, which has nothing to do with this approach. Namely, from this analysis we have extracted the notion of fibration together with a point of view (cf. (10.4.2)), and we claim that this is enough to recapture all the richness and complexity of naive category theory, and also to clarify many aspects of topos theory. Although some hints are given, there is no real justification of this claim.

(0.9) This should not be surprising. Such a claim *cannot* be justified on the basis of the analysis itself because it "speaks about" the analysis, and "says" that it has been thorough enough. The only "proof" of such a claim can be found in actual work with fibrations, showing that the major notions and results of naive category theory do translate in the context of fibrations, and how they translate. This work can be found elsewhere (see the bibliography).

However, as mentioned in (0.7)(iv), the analysis given in this paper, because of its possibility of "prediction", is a great help in this work, and in particular permits us to avoid some very tricky mistakes related to the equality of objects (cf. (12.5)).

(0.10) In §§1–5 we gather some basic facts from naive category theory and begin to find out what our "categories" will have to look like.

The most important parts are §§6 and 7, where we introduce the notion of definability which will eventually play the role of a "set-free analog" of the comprehension scheme, and §8, where a very careful study of equality leads to the very strange conclusion that in our "categories" the equality of two objects may not have any meanings, or may have many.

In §§9–11 the analysis is concluded by saying what our "categories" are and what kind of mathematics (apart from foundations) they cover.

In 1970 I realised the possibility of doing all of naive category theory, without sets, in the context of fibrations, and started work on proving that claim (cf. (0.9)). After many results in that direction, and on the basis of these results, I proposed to Jean Celeyrette as an important test that he prove the "fibered analogs" of Kan's theorem and the adjoint functor theorem. One of the major steps in that direction was to clarify the property for a fibration of being locally small. All this was completed in 1974, and I gave a series of lectures in Montréal containing all of Celeyrette's results, and many others.

In 1978 another approach was proposed [John-Par]. Thus I have included an appendix where the two approaches are compared, from two points of view: mathematics and foundations.

I have added a short annotated glossary. Since it assumes some familiarity with fibered categories, no reference is made to it in the text. It shows how some important notions, apparently vague in the paper, can be made precise, and explains a bit more what I mean by a "category without equality of objects".

§1. The first observations about naive category theory. Open any text about categories, whether it be a short paper or a book, elementary or specialized, and you cannot help observing a few things.

(1.1) The ubiquity of commutative diagrams, even if they are not displayed.

(1.2) The frequency of statements of the form: two objects X and Y of a category **C**, constructed in such and such manner, are isomorphic.

(1.3) The absence of statements of the form: the objects X and Y are equal.

There will also be numerous assumptions, or conclusions, about a category **C**, of the following types.

(1.4) **C** is small, or **C** is *locally small* (i.e. for each pair X, Y of objects of **C** the maps from X to Y form a set $\mathrm{Hom}(X, Y)$).

(1.5) **C** admits infinite products (or other sorts of infinite limits).

The following constructions will be frequently used, and they will be assumed to be obvious and harmless.

(1.6) If **C** is locally small, we can construct for each pair X, Y of objects of **C** the set $\mathrm{Mono}(X, Y)$ of all monomorphisms from X to Y.

(1.7) If $(X_i)_{i \in I}$ is a family of objects of a category **C**, and $u\colon J \to I$ is a set map, we can construct the family $(Y_j)_{j \in J}$ defined by $Y_j = X_{u(j)}$ $(j \in J)$.

(1.8) For each family $(f_i\colon X_i \to X'_i)_{i \in I}$ of maps of **C**, we can construct the subset J of I defined by $J = \{i \in I \mid f_i \text{ is an isomorphism}\}$. (And of course "isomorphism" can be replaced by "monomorphism" or "epimorphism".)

Statements of the following type will be considered as obvious and used without even being thought worth mentioning.

(1.9) Let **C**$'$ be a subcategory of **C**. If **C** is small, or locally small, so is **C**$'$.

(1.10) The Yoneda embedding of a category **C** into the category $\hat{\mathbf{C}}$ of functors from the dual $\mathbf{C}^{\mathrm{op}}$ of **C** into the category **Ens** of sets will play a major role. As we know that $\hat{\mathbf{C}}$ poses some logical problems, to be on the safe side there will usually be assumptions of some type: **C** is small, or we are not looking really at **Ens** but at the sets of a given, usually unspecified, universe **U**. Yet everyone knows that as soon as **U** is big enough, the properties of this functor (e.g. it is full and faithful) do not depend on **U**, and are "purely formal".

§2. The usual foundations. (2.1) There is of course a very simple first-order theory of categories, the models of which are "small categories". But it excludes or trivialises such fundamental notions as categories with infinite limits of various sorts, and thus it is inadequate.

(2.2) There is also a Bernays type distinction between sets and classes, but it does not allow arbitrary functor categories, which we would very much like to have.

More elaborate versions have been proposed by logicians, but they have become so utterly complicated, and so far from the actual way we think about, and manipulate, categories and their relationship to sets, as to be totally inadequate.

(2.3) The framework of universes, adopted say in (SGA), is perfectly consistent, assuming a strengthening of ZF, but the remarks at the end of (1.10) show that it is not quite satisfactory, and again does not reflect the way we work with categories.

(2.4) The frameworks described in (2.2) and (2.3), apart from their inadequacy, have a very unpleasant common feature: they are based on "set theories" at least as strong as ZF, thus excluding the possibility of taking as "sets" the objects of an elementary topos, the importance of which need not be emphasized.

§3. The basic notions of our formal system. (3.1) In view of the importance of smallness conditions in category theory, we shall take as primitive notions in our framework those of "sets" and maps between "sets", with composition; thus we shall start with a basic category **B**, which we will think about as being **Ens**, but on which, for the moment, we make no further assumption. The objects of **B** will be denoted by I, $J, \ldots$, and its maps by $u\colon J \to I$, $v\colon K \to J, \ldots$.

(3.2) Now since our "sets" $I, J, \ldots$ need not have elements, there is no way to define the notions of family $(X_i)_{i \in I}$ of objects of a category **C**, or the family $(f_i\colon X_i \to Y_i)_{i \in I}$ of maps. So these again will have to be taken as primitive notions. Thus, once our category of "sets" **B** is chosen, the notion of "category" should carry at least the following information: for each object I of **B**, a category (in the usual sense) $C(I)$, called the *fiber over* I, whose objects and maps should be thought of as "families" of objects and maps of our "category", indexed by I.

(3.3) However, this information is not sufficient, as we shall now see: In the "naive case", i.e. when $\mathbf{B} = \mathbf{Ens}$ and "families" are real families of objects and maps of an ordinary category **C**, the collection of all such families, indexed by variable sets $I \in \mathbf{Ens}$, admits an extra structure. If $X = (X_i)_{i \in I}$ and $Y = (Y_j)_{j \in J}$ are two families of objects of **C**, we can define an *external map* $\phi\colon Y \to X$ to consist of a pair (u, f), where $u\colon J \to I$ is a set mapping, and $f = (f_j\colon Y_i \to X_{u(j)})_{j \in J}$ is a family of maps of **C**. If $Z = (Z_k)_{k \in K}$ and $\psi = (v, g)\colon Z \to Y$, we can obviously define the composite $\phi\psi = (w, h)\colon Z \to X$ by $w = uv$ and

$$h = (h_k\colon Z_k \xrightarrow[g_k]{} Y_{v(k)} \xrightarrow[f_{v(k)}]{} X_{uv(k)})_{k \in K}.$$

Thus we obtain a category **Fam(C)**, called the *category of families of* **C**. It is equipped with a projection functor $p_{\mathbf{C}}\colon \mathbf{Fam(C)} \to \mathbf{Ens}$, defined by

$$p_{\mathbf{C}}((X_i)_{i \in I}) = I \qquad \text{and} \qquad p_{\mathbf{C}}((u, f)) = u.$$

An internal map $f = (f_i\colon X_i \to X'_i)_{i \in I}$, for any set I, can be identified with the external map (id_I, f), and a map ϕ of **Fam(C)** is of this form iff $p_{\mathbf{C}}(\phi)$ is an identity. Hence the knowledge of the category **Fam(C)**, together with the functor $p_{\mathbf{C}}$, entirely determines, up to trivial isomorphisms, the structure of all the fibers, as required in (3.2).

(3.4) Now if we try to replace **Ens** by an arbitrary category **B**, even if we are given for each object I of **B** a category $C(I)$, there is no natural way to define external maps between objects Y of $C(J)$ and X of $C(I)$, for $I \neq J$, in order to get a total category containing all the fibers, analogous to $p_{\mathbf{C}}$.

(3.5) Yet, even if they are not actually mentioned, the knowledge of the total category **Fam(C)** and the functor $p_{\mathbf{C}}$ is implicit in considerations about the category **C** in naive category theory, and plays an important role. For example, the best way to express the fact that **C** has arbitrary coproducts, and to get all the naturality properties of the symbol $\coprod_{i \in I} X_i$ with respect to both the set I and the family $(X_i)_{i \in I}$ is to say that the diagonal functor $\eta_{\mathbf{C}}: \mathbf{C} \to \mathbf{Fam}(\mathbf{C})$ (which assigns to each object or map of **C** the obvious families, indexed by 1, of objects or maps of **C**) has a left adjoint: $\mathbf{Fam}(\mathbf{C}) \to \mathbf{C}$; $(X_i)_{i \in I} \mapsto \coprod_{i \in I} X_i$.

(3.6) Now, in foundations, we cannot leave implicit such a major notion, and since, by (3.4), it cannot be reduced to just **B** and the fibers $C(I)$, it has either to be given as basic, or to be constructible from the basic notions. Hence, whatever our definition of a "category" will eventually be, it will have to carry at least the following information: two categories **B** and **F**, playing the role of **Ens** and **Fam(C)**, and a functor $p: \mathbf{F} \to \mathbf{B}$ playing the role of $p_{\mathbf{C}}$.

§4. What will the theory of "categories" look like? (4.1) When we think about, or work with, a category **C**, many connexions with sets are implicitly assumed. The notion of "families" is only one of them, giving rise to structure, namely the functor $p_{\mathbf{C}}$ involving both **C** and sets. We have abstracted this sort of connexion in terms of a functor p, but it may turn out that other types of connexions will have to be made explicit, each of them giving rise to an abstract structure in our framework. Anticipating on (10.2) let us say that, apart from a few cases where an extra structure very closely related to p is required, all the structure we need will be contained in p. However, this functor will have to possess certain properties if it is to play the role of the functor $p_{\mathbf{C}}: \mathbf{Fam}(\mathbf{C}) \to \mathbf{Ens}$. We will use $p_{\mathbf{C}}$ as a guide to find relevant assumptions to make about p, but we can already predict a few things about them.

(4.2) Some of them are essential, and will have to be made in all cases, since they reflect the construction of **Fam(C)** and $p_{\mathbf{C}}$ or, more profoundly, the very idea of "families". Though we still have to find what they are (cf. §9), we can introduce a temporary terminology.

If $p: \mathbf{F} \to \mathbf{B}$ is a functor satisfying these unknown assumptions, we will say that p is a "category" defined over the "sets" **B**, or a **B**-"category". Thus, rephrasing (3.5), in naive category theory we talk about a category **C**, but what we really use is the "category" $p_{\mathbf{C}}$, of which **C** is but a small part.

(4.3) Some assumptions will involve only **B**. They express the strength we need (or want) our notion of "sets" to have. For example the assumption: "**B** is an elementary topos with natural numbers object" should be enough for almost all purposes. It will actually be much too strong in many cases, and a lot can be done just assuming that **B** has pull-backs.

(4.4) Some assumptions, no matter what is assumed about **B**, will have to involve the whole structure given by p. They reflect properties of the relationship existing between our "category" and our "sets" ("smallness" conditions, "infinite" limits, etc....).

(4.5) In principle we even have a method for finding many relevant assumptions on p. Take any "important" property $\mathscr{P}$ of a category **C**. In most cases it will be possible to find a property $\mathscr{P}'$ of functors such that **C** satisfies $\mathscr{P}$ iff the functor $p_{\mathbf{C}}$ satisfies $\mathscr{P}'$. We will then have a translation of $\mathscr{P}$ into our language by declaring that a "category" p satisfies $\mathscr{P}$ iff the functor p satisfies $\mathscr{P}'$.

(4.6) We must insist on the fact that, since **B** is no longer **Ens**, $\mathscr{P}'$ has to be expressed, even in the case of $p_{\mathbf{C}}$, without reference to set theory, say in the language of the first-order theory of categories and functors. For properties of type (4.3) topos theory has taught us how to do so. For those of type (4.4), the main difficulty is to make explicit the countless relations between categories and sets which are implicitly assumed in naive category theory.

This can be very tricky because these assumptions are so deeply rooted that we do not think of them as being assumptions, and most of the time we are not even aware of their existence (cf. (5.3)). Thus, apart from the significant generalization from **Ens** to **B**, our setting will help us, even in the case of naive category theory, where **B** = **Ens**, to discover these assumptions, to understand their real meaning, and to describe their syntactic structure. This will in turn make it easy for us to extend many results from the case **B** = **Ens**, to **B** either arbitrary, or subject to some conditions, the reason of which shall be understood.

§5. Some "paradoxes" in category theory. From the beginning, categories were introduced to describe some entities too big to be sets (all groups, rings, etc....). But as soon as *some* categories are not sets, we have to conclude that:

(5.1) *Categories need not be sets* (even if some of them are).

From the example of big categories it is very easy to draw a false conclusion, and it has indeed been implicitly drawn, namely, that the only reason why a category could fail to be a set is that it is too big.

We claim that as soon as we admit some categories which are not sets it may very well happen, and it is indeed bound to happen, that:

(5.2) *There exists a small category* **C** (i.e. which is a set) *admitting a subcategory* **C**′ *which is not small.*

We can even strengthen the paradox by each of the two requirements "**C**′ is not locally small" or "**C** is finite and discrete".

Even more striking than (5.2), where one could argue about the definition of a subcategory of a small category, we may have:

(5.3) *There exist a locally small category* **C**, *and two objects* X, Y *of* **C**, *such that the monomorphisms from* X *to* Y *do not form a set.*

Now, if (5.3) is true, it makes (1.6) meaningless. But even if it is false, it makes us aware of the fact that, by merely writing (1.6), we make very strong assumptions, that we will now analyse. To refute (5.3):

Let $\mathrm{Mono}\{X, Y\}$ be $\{f \in \mathrm{Hom}(X, Y) \mid f$ is a monomorphism of $\mathbf{C}\}$. Then obviously the comprehension scheme (C.S.), and the fact that $\mathrm{Hom}(H, Y)$ is a set since we assume **C** locally small, guarantee that $\mathrm{Mono}(X, Y)$ is a set. However, C.S. only says that, if S is a set and ϕ a formula of set theory, one can construct the subset T of S defined by $T = \{s \in S \mid \phi(s)\}$.

But, even if **C** is locally small, there is no reason why "f is a monomorphism of **C**" should be (equivalent to) a formula of set theory.

Of course examples of this type can be multiplied by replacing "mono" by "epi" or "iso", or various other categorical notions.

Again if **C** is locally small, and X' and Y' are objects of a subcategory **C**′ of **C**, the set $\mathrm{Hom}_{\mathbf{C}'}(X', Y')$ should be defined, in violation of C.S., as $\{f \in \mathrm{Hom}_{\mathbf{C}}(X', Y') \mid f$ is a map of $\mathbf{C}'\}$. Thus **C**′ need not be locally small.

(5.4) At this stage one can make the following objections to the "strange" statements of type (5.2) or (5.3).

(i) They arise out of "logical hair splitting", perhaps relevant in axiomatic set theory, but certainly of no importance in category theory.

(ii) They can easily be disposed of with a distinction between sets and classes, e.g. à la Bernays, satisfying the axiom: "each subclass of a set is a set".

These objections are *quite serious* and deserve a closer investigation. Now (ii) implies a strengthening of ZF which has been rejected in (2.2) as inadequate, and, more important, for mathematical reasons in (2.4) since it excludes the possibility of taking **B** to be an elementary topos. As for (i), topos theory shows that there is no frontier between logical considerations about sets and category theory, Moreover, since we are speaking of foundations we have to be able "to split all those hairs that might pose problems".

There still remains one objection, psychological, but all the more important: we do not want to be trapped into some "strange", unpredictable, type of mathematics. Naive category theory is all but strange, and its foundations should reflect its "naturality". At this stage, the only answer we can make is: we have to carry out the analysis completely and "hope" that all the strange notions will eventually fit in a perfectly coherent, natural and simple pattern. (This will indeed be the case.) Yet we can be comforted in this hope if we remark that, on the basis of a superficial analysis, "strange" things do happen in toposes, e.g. there exist "sets" without "elements" which are not "the empty set".

In the rest of the paper many objections could be made. We will ignore those which are only based on "strangeness". However, they "force" the following remarks.

(5.5) If, already with $\mathbf{B} = \mathbf{Ens}$, as soon as categories are not required to be sets we run into apparent paradoxes such as (5.2) or (5.3), we have to expect that, for an arbitrary **B** (e.g. a topos), hand-waving about such problems will no longer be possible, and they will have to be treated seriously.

(5.6) Being quite liberal about accepting strange notions has a dialectical counterpart: being very strict about accepting anything as "obvious". Thus we will not shy from making explicit some apparent trivialities. In questions of foundations the difficult problems usually arise from the situations which are the most trivial in the naive approach (see the discussion about equality in §8).

§6. Formal definability. Consider the following question, which is a prototype of many others.

(6.1) Let **C** be a locally small category. What assumptions do we need to define, for all objects X and Y of **C**, the set $\mathrm{Mono}(X, Y)$?

This is *not* a precise question, and the answer depends on what we call category and what we call sets.

We will first make it precise, and give a partial answer, using rather strong notions of sets and classes, then try to see how much these notions can be weakened.

Some rudiments of axiomatic set theory will be needed. I apologize for their triviality. Besides (5.6), my excuses will be the following:

(i) The terminology I introduce will be adaptable to more general situations, and will even *suggest* meaningful generalizations.

(ii) After many years of work with fibered categories, and although lots of things had come out very nicely, I had the feeling that some basic idea was missing. And it was actually in trying to explain to nonmathematicians the difference between naive set theory and axiomatic set theory, and especially the comprehension scheme, that I found the answer, in terms of definability (cf. §7).

(6.2) By *a set theory*, we mean any theory **T** written in the language of ZF and satisfying *at least* the extensionality axiom E and the comprehension scheme C.S. (You can take ZF if that makes you feel safer.)

Let $\mathcal{M}$ be a model of such a theory. The elements of $\mathcal{M}$ will be called *sets* and denoted by $S, T, \ldots$. The *formal* membership and equality for sets will be denoted by $S \mathrel{\bar{\in}} T$ and $S \equiv T$.

The "meta-sets" of the universe of discourse **U** where the model $\mathcal{M}$ is taken will be called *classes* and denoted by $\mathcal{S}, \mathcal{C}, \ldots$. Membership, equality, inclusion, intersection, etc.... in **U** are denoted by the usual notations $\in, =, \subset, \cap, \ldots$. In particular $\{\ldots | \ldots\}$ will denote a class.

A subclass $\mathcal{S}$ of $\mathcal{M}$ is called *representable* if there is a set S, called a *representative of* $\mathcal{S}$, such that for all $T \in \mathcal{M}$ we have $T \in \mathcal{S}$ iff $T \mathrel{\bar{\in}} S$.

We have the *Yoneda map* which assigns to each set S the representable class

$$\hat{S} = \{T \in \mathcal{M} \mid T \mathrel{\bar{\in}} S\}.$$

A subclass $\mathcal{S}$ of $\mathcal{M}$ is called *formally definable* if there is a formula $\phi(x)$, with parameters in $\mathcal{M}$, such that, for all $T \in \mathcal{M}$, we have $T \in \mathcal{S}$ iff $\mathcal{M} \models \phi(T)$.

The extensionality axiom and comprehension scheme read thus:

(E) For all sets S and T, $\hat{S} = \hat{T}$ iff $S \equiv T$.

(C.S.) If $\mathcal{S}$ is representable and $\mathcal{C}$ formally definable, then $\mathcal{S} \cap \mathcal{C}$ is representable.

(6.3) We can now give some obvious definitions.

A category **C** is defined by a class $\mathcal{C}_0$ of objects, a class $\mathcal{C}_1$ of maps, a composition and identities, as usual, with the precise meaning of class. If $\mathcal{C}_0$ and $\mathcal{C}_1$ are representable, we say that **C** is *small*. If for all $X, Y \in \mathcal{C}_0$ the class $\mathcal{H}om(X, Y)$ of all maps from X to Y is representable we say that **C** is *locally small* and denote by $\mathrm{Hom}(X, Y)$ a representative of this class. Notice that **C** locally small $\Rightarrow \mathcal{C}_1 \subset \mathcal{M}$. Thus, local smallness is a very strong property.

A partial answer to (6.1) is now obviously given by:

(6.4) If **C** is locally small, and the class $\mathcal{M}ono(\mathbf{C})$ of all monomorphisms of **C** is formally definable, then, for all $X, Y \in \mathcal{C}$, the class $\mathcal{M}ono(X, Y) = \mathcal{H}om(X, Y) \cap \mathcal{M}ono(\mathbf{C})$ is representable, and we can call $\mathrm{Mono}(X, Y)$ any set representative of this class.

In the same vein, with obvious definitions, we have:

Let **C**$'$ *be a formally definable subcategory of* **C**. *If* **C** *is small or locally small, so is* **C**$'$.

We have now to worry about the question:

(6.5) *How strong must our set theory be*?

We also have to remember that whatever answer or answers we give to this question, with these very specific definitions of categories, sets, etc...., will have to be reflected in the abstract setting of "categories" (cf. (4.2)). And if this setting is adequate, this reflection should permit us to predict partially how "categories" ought to behave.

(6.5.1) So far, using only E and C.S. for our sets, we have been able to define small or locally small categories; and many more definitions could be given just by requiring that some classes be representable. But not much can be proved from these definitions if our set theory **T** is too weak. This should be (and will actually be) reflected by the fact that we can define for an arbitrary **B**-"category" many properties of type (4.4), e.g. smallness, local smallness, etc...., with no assumptions or very mild assumptions on **B**. But in order to prove theorems, we will have to assume some properties of type (4.3).

(6.5.2) Notice however that, since "big" categories are just "small categories with respect to **U**", many properties of big categories are available as mere reflections of properties of **U** regardless of the power of **T**. The reflection of this would be that *the "totality" of all* **B**-*"categories" must be equipped with a very rich structure, without any assumption on* **B**, coming from the "outside" universe **U**.

(6.5.3) We could, and should, go a bit further and axiomatize the universe **U**, which, at the naive level, would amount to serious study of the question: *What do we need about our "classes"*? We will not do it for the sake of brevity.

(6.5.4) With E and C.S. alone, no matter what strong assumptions we make about **U**, we will not be able to define the category **Ens** of sets. We know its class of objects, namely $\mathscr{M}$, but to get the maps we must strengthen **T** a little. (This is classically achieved by much too strong requirements, the axioms of pairing and power set, which force **Ens** to be an elementary topos, but it could be done more economically.)

However, if **Ens** is to exist at all as a category it will automatically have some strong properties. For example if S and T are sets, $\hat{S} \cap \hat{T}$ is representable because of C.S.; hence finite intersections of subobjects will exist in **Ens**. Again, if **T** is strong enough to allow the existence of **Ens**, this category will play a fundamental role for all smallness properties. For example, any locally small category **C** will come equipped with a canonical functor for Hom: $\mathbf{C}^{\mathrm{op}} \times \mathbf{C} \to \mathbf{Ens}$.

We insist on the fact that this is not contained in the definition of a locally small category, and has to be proved. We recall that the notion of locally small category can be, and has been, defined even when **Ens** does not exist!

(6.5.5) On the basis of (6.5.4) we can "predict" that some mild assumptions on the category **B** will be necessary, and sufficient, to define a fundamental "category" $p(\mathbf{B})$ playing among all "categories" over **B** the same role as **Ens** plays among all categories in naive category theory. Moreover $p(\mathbf{B})$ will have very strong properties, as a **B**-"category", and will play a major role for all smallness considerations.

§7. Definability and representability, or how to get rid of set theories. Even if we have gained significant generality by cutting down the axioms of ZF to E and C.S., with possibly some mild additions, there still remains the fact that we use the language of set theory. For example our answer to (6.1) depended essentially on $\bar{\in}$, via formal definability. For many reasons, e.g. toposes, we would like not to depend on that language.

This can be achieved by introducing a new essential notion, first using a set theory, and then seeing how it can be translated into the language of category theory.

We keep the notations of §6 for **T**, $\mathscr{M}$, sets, classes, etc.....

(7.1) A class $\mathscr{S}$ is called *definable* (semantically in $\mathscr{M}$) iff for each representable class $\mathscr{C}$ the class $\mathscr{C} \cap \mathscr{S}$ is representable. For each set T we can then construct a set $T^*(\mathscr{S})$, unique up to formal equality, such that $T^*(\mathscr{S}) = \{S \bar{\in} T \mid S \in \mathscr{S}\} = \mathscr{S} \cap \hat{T}$.

This definition calls for a few comments.

(7.2) Obviously just assuming E and C.S. we have

$$\mathscr{S} \text{ formally definable} \Rightarrow \mathscr{S} \text{ definable.}$$

But, even assuming ZF and $\mathscr{S} \subset \mathscr{M}$, the converse need not be true.

(7.3) This notion of definability is not syntactic but belongs to model theory. I think that even in ZF it should deserve some study. For example, are there models $\mathscr{M}$ of ZF such that $\mathscr{S} \subset \mathscr{M} \Rightarrow \mathscr{S}$ definable?

(7.4) The notion of definability admits many variations. The most important is the "relative version" obtained as follows:

A subclass $\mathscr{S}'$ of $\mathscr{S}$ is *definable in* $\mathscr{S}$ iff for each representable class $\mathscr{C}$ contained in $\mathscr{S}$, the class $\mathscr{C} \cap \mathscr{S}'$ is representable.

That this is a relative notion is readily seen by noting that $\mathscr{S}$ is always definable in $\mathscr{S}$, even if $\mathscr{S}$ fails to be definable.

(7.5) Of course, now a more general answer to (6.1) is obtained by just replacing in (6.4) "formally definable" by "definable in $\mathscr{C}_1$".

(7.6) Since definability is completely known as soon as we know which classes are representable, the role of $\bar{\in}$ is now reduced to the definition of representable classes.

Thus as soon as we have, via $\bar{\in}$ or any other means, a "reasonable" notion of representability, it should carry along with it a notion of definability almost as important.

But now, for any category, we do have a notion of representability that is of major importance—the usual representability of "set"-valued functors (see the last sentence of (1.10)). Thus we have to expect that:

(7.7) *For* **B**-*"categories", some notion of definability closely related to the representability of functors will be present.* And because of (7.5) and the numerous variations it clearly admits, *definability will play a major role in smallness considerations.*

§8. Equality in category theory. This is, by far, the most subtle and strange notion. A syntactic analysis of the type carried in §6 would require a formalization of **U**, especially the equality in **U** (cf. (6.5.3)). Thus our approach will be mainly descriptive: How is equality really used in actual work in naive category theory?

We will analyse this question in terms of the sets and classes defined in (6.2), but try to get answers adaptable to the abstract situation of "categories".

The discussion will be based on (1.1), (1.2) and (1.3).

(8.1) Since the very definition of categories involves commutative diagrams, (1.1) is hardly surprising. But to say that a diagram commutes means that certain parallel pairs of maps (f, f') (i.e. such that f and f' have same domain and codomain) satisfy $f = f'$. But this really tells us two things:

(i) Even though we do not know what requirements have to be made about our universe **U** of classes, they must, at least, be such that, for any category **C**, the parallel pairs of maps of **C** form a class $\mathscr{P}ar(\mathbf{C})$, and the parallel pairs such that $f = f'$, whatever equality means in **U**, form a subclass $\mathscr{E}q(\mathbf{C})$ of $\mathscr{P}ar(\mathbf{C})$.

(ii) If we want to prove anything about this equality by means of our sets, we will very quickly have to assume some relations between these classes and our sets, e.g.

for a family $(f_i, f_i')_{i \in I}$ of parallel pairs indexed by the set I, to be able to use the subset J of I having as elements the i's such that $f_i = f_i'$. Thus, the class $\mathscr{E}q(\mathbf{C})$ will have to be definable in $\mathscr{P}ar(\mathbf{C})$. When $\mathbf{C}$ satisfies this property we shall say that *equality* (of parallel pairs) *is definable in* $\mathbf{C}$.

(8.2) A closer inspection of statements of form (1.2) shows that what they really say is that a specific map $f: X \to Y$, either given or uniquely determined once X and Y have been "constructed", is an isomorphism. For brevity f is frequently omitted, but this is no longer possible in a careful analysis. And again this tells us two things:

(i) The universe must be such that, for any category $\mathbf{C}$, the isomorphisms of $\mathbf{C}$ form a class $\mathscr{I}so(\mathbf{C})$.

(ii) To talk in terms of our sets about the isomorphisms of $\mathbf{C}$ we will have to assume that isomorphisms are definable in $\mathbf{C}$, i.e. $\mathscr{I}so(\mathbf{C})$ is definable in $\mathscr{C}_1$.

(8.3) Thus we should expect that in most "categories" equality and isomorphisms will be definable (cf. (10.4.3)).

Before we turn to (1.3), which is the most difficult, some objections not based on strangeness could be made:

(8.4) If such properties as definability of equality or isomorphisms are so basic, how come they do not appear in naive category theory? In fact much stronger properties are implicitly used all the time: e.g. $\mathbf{U}$ is a universe in the technical sense of [SGA], $\mathscr{M}$ is a universe element of $\mathbf{U}$ and $\bar{\in}$, $\equiv$ as well as $\in$ and $=$ are restrictions of the same membership and equality relation coming from a still bigger universe $\mathbf{V}$ satisfying all of ZF. They also occur in countless constructions of the type: "Let $\mathbf{C}$ be a locally small category, (f, f') a pair of maps from X to Y, and $Z \in \mathscr{C}_0$. Consider the set of all maps $g: Y \to Z$ such that $gf' = gf$...".

(8.5) There are "many" important categorical notions. Will not the final setting be very complicated, and therefore inadequate, if it compels us at each moment to make ad hoc definability assumptions?

It need not be so, and the general situation will come out as follows: Assuming for a $\mathbf{B}$-"category" p that a few basic categorical notions are definable, we will be able to prove, if $\mathbf{B}$ is rich enough (e.g. a topos, though this is often too much) that many more notions are definable for p, and also for many "categories" constructed from p.

The existence of such results is "plausible" on the basis of our class-set analysis, because what they really say is that if "a few" classes, determined by a category $\mathbf{C}$, are definable, and if we can perform "many" constructions with our sets, then many classes will have to be definable. Moreover this analysis will give us strong indications on what should be true about definability for "categories" and how to prove it, if it is carried out carefully and thoroughly.

(8.6) What about the equality of objects? We claim that (1.3) is the reflection, in the written corpus of naive category theory, of the following facts:

(i) *There is no categorical way to detect, for two objects X and Y of an abstract category* $\mathbf{C}$, *if they are equal.*

(ii) *Moreover we do not care about such a question, which should in general be meaningless, if no other information is given about* $\mathbf{C}$.

(8.7) Categorical properties are preserved and reflected by equivalences of categories. However equivalences *do not* reflect the equality of objects, hence (i). The same argument goes for those (very few) properties of diagrams which are not reflected by equivalence of categories, e.g. the property for a map to be an identity, or the equality of two *arbitrary* arrows, not parallel.

Hence, in contrast with (8.1) and (8.2) there is no reason to assume, for an arbitrary category **C**, any of the following properties:

(i) Pairs of equal objects are definable in all pairs of objects.

(ii) Identity maps are definable in all maps.

(iii) Pairs of equal maps are definable in all pairs of maps.

That we do not care about the meaning of "$X = Y$" is seen as follows. Suppose we know that two objects X and Y are "equal". Unless this equality is some "god-given" identity, by performing very trivial constructions, say $X \times X$ and $Y \times Y$, we get two objects defined up to a unique isomorphism, for which we can no longer be sure that they are still equal.

(8.8) Important categorical constructions confirm (8.1), (8.2) and (8.6). In algebra we weaken equality by taking congruence relations, which equalize pairs of elements. There is no significant construction in category theory which forces objects to become equal. But there are two major notions of "congruences" in category theory, reflecting (8.2) and (8.1):

(i) Categories of fractions, where some given maps are forced to become isomorphisms (not just some pairs of objects to become isomorphic).

(ii) Quotient categories where parallel pairs of maps are identified, without altering the objects (e.g. the homotopy category).

(8.9) We are now compelled (and ready) to make some "heretic" statements. If equality of objects may be undefinable, or if we do not care if it is, there is no reason, according to (0.4)(ii), to reject those "categories", if any, which do not admit any "good notion" of equality of objects, or again those which admit many such notions. Thus:

(8.9.1) Equality of objects should be an extra structure, which a "category" p may or may not admit; such a structure, if any, need not be unique.

A pair consisting of p and one such structure shall be called a *"category" with equality of objects.*

The question, "is equality of objects definable?" should make sense only when p is equipped with such a structure.

Since by now we are doomed to the stake, let us be heretic all the way. If a "category" can admit two such structures, that paragon of all "functors", the identity, has to destroy at least one of them. Thus:

(8.9.2) The "functors" between "categories" with equality of objects need not preserve this equality. We may have to select those which do, called *equality preserving "functors"*.

(8.9.3) There is no reason to suppose that our "category" of "sets" $p(\mathbf{B})$, when it exists (cf. (6.5.5)), should admit such a structure.

In fact many reasons plead for the contrary; we mention a few.

(i) We want to consider different notions of "sets", as general as possible.

(ii) Even in the restricted context of set theories based on membership (cf. §6) we are confronted with two types of equalities for sets: $T \equiv S$, and the external equality $T = S$ in **U**. The second is the only one meaningful for classes, yet it is not definable in **Ens**.

(8.10) If we carried this analysis a bit further, we could "predict" many more things about the equality of objects. We shall state a few as exercises, and treat another one with more detail in (8.11).

(i) Some "categories" should have a unique equality of objects, e.g. the discrete, or more generally the ordered "categories". (Isomorphisms can be detected, but in these cases they are unique and are identities.)

(ii) Some should come equipped with a canonical equality of objects, e.g. the small "categories" and, more surprisingly, the functor "categories" even between "categories" without equality of objects. (In functor categories we care about, and know, what the objects are.)

(iii) It should be possible to adjoin freely an equality of objects to any "category".

(8.11) The category of **B**-"categories" need not have all pull-backs. The argument goes as follows: let $F_i: \mathbf{C}_i \to \mathbf{C}$ $(i = 1, 2)$ be ordinary functors. The pull-back $\mathbf{C}_1 \times_{\mathbf{C}} \mathbf{C}_2 = \mathbf{P}$ has as objects pairs (X_1, X_2) of objects such that $F_1 X_1 = F_2 X_2$. Thus, to be able to define **P** we have to know the meaning of $X = Y$ in **C**, at least for some pairs (X, Y) of objects. (This hints moreover that some pull-backs should exist if the "functors" satisfy good compatibility conditions.)

Now, this simple-minded, nonmathematical "argument" contradicts precise statements made in [Gi] and [Par-Sch]. In fact, *these statements are false*, and easy counterexamples, of the most traditional mathematical type, can be given. This shows that the idea of a "category" without equality of objects can have some mathematical significance.

(8.12) Before we proceed, we beg the reader to stop and consider the following question: Naive category theory is not elementary. We talk about properties of categories (local smallness, infinite products, well-poweredness, etc. ...) using the language of sets and the whole strength of ZF. What would all this mean if, instead of sets, we used some Boolean valued model of a set theory? And in case some meaning can be given, would not the equality of objects cause some problems? in particular with the notion of pull-back of categories?

§9. What is a "category", or a "category" with equality of objects? At this stage a mathematical answer would be very easy—much easier indeed than the analysis we still have to carry out. But that would violate (0.4)(ii). Moreover we have reached such "unheard of" notions as "categories" without equality of objects on the "experimental" basis of observation, followed by analysis of the observation. But how do we know if the observation has not overlooked some vital phenomenon, or if the analysis is more than a linguistic game? Or again, we may have been too ambitious in our search for greater generality and such notions may not exist, or if they do they may turn out to be artificial monsters, totally useless for mathematical purposes.

So, we have to find properties of type (4.2) for a functor $p: \mathbf{F} \to \mathbf{B}$, by analysis of the example $p_{\mathbf{C}}$: **Fam(C)** $\to$ **Ens** (cf. (3.3) and (4.6)).

(9.1) Let $X = (X_i)_{i \in I}$ be a family of objects of **C**, and $u: J \to I$ be a set map. We can define a family $Y = (Y_j)_{j \in J}$ by $Y_j = X_{u(j)}$ $(j \in J)$. This seems obvious, and harmless, *but it is not*. Remember that we do not care about the meaning of $X_1 = X_2$ in **C**, and that in our "categories" this may have no meaning at all; hence we may not know what $Y_j = X_{u(j)}$ means. On the other hand we cannot altogether forget about this definition, since the possibility of re-indexing is one of the most basic features of the very idea of families, which we are trying to formalize.

Thus, all we can require in the abstract case $p\colon \mathbf{F} \to \mathbf{B}$, is the following:

(i) For each object X of $\mathbf{F}$ and map $u\colon J \to pX = I$ in $\mathbf{B}$, there exists at least one $Y \in \mathbf{F}$ such that $pY = J$, playing "the same role" as the family $(Y_j = X_{u(j)})_{j\in J}$.

(ii) If p is equipped with an equality of objects, using this extra structure (8.9.1), we can select among all such possible Y's, a specific one which "behaves nicely" with respect to this equality.

(9.2) All that remains to do is, in the case of $p_{\mathbf{C}}$, to find a property of $(X_{\mu(j)})_{j\in J}$ defined only in terms of the functor $p_{\mathbf{C}}$, independent of the equality of objects in $\mathbf{C}$, characterizing this family up to a unique isomorphism. But we can define an external map in the sense of (3.3), $\phi_{(u,X)} = \phi\colon Y \to X$, by $\phi = (u, \mathrm{id})$, where id denotes the family $(\mathrm{id}_{Y_j}\colon Y_j = X_{u(j)})_{j\in J}$. The pair (Y, ϕ) is characterized, up to a unique isomorphism, by the following property:

For each map $\psi = (v, g)\colon (Z_k) \to (X_i) \in \mathbf{Fam(C)}$ and each factorization $v = uv'$ in **Ens**, there exists a unique factorization $\psi = \phi\psi'$ in **Fam(C)** such that ψ' can be written as (v', g'), i.e. $p_{\mathbf{C}}(\psi') = v'$.

Moreover, if we use the equality of objects in $\mathbf{C}$ to define (Y, ϕ), as $((X_{u(j)}), \phi_{(u,X)})$, these choices are functorial, i.e. satisfy the conditions

$$\phi_{(\mathrm{id}_I, X)} = \mathrm{id}_X \quad \text{and} \quad \phi_{(uv,X)} = \phi_{(u,X)} \circ \phi_{(v,Y)}$$

(for $v\colon K \to J$). We can extend (9.2) to $p\colon \mathbf{F} \to \mathbf{B}$ via the following definitions, due to Grothendieck.

(9.3)(i) p is a *fibration* (or *fibered category*) *over* $\mathbf{B}$ iff for each $X \in \mathbf{F}$ and $u\colon J \to I = p(X) \in \mathbf{B}$, there exists a map $\phi\colon Y \to X \in \mathbf{F}$ satisfying "$p\phi = u$, and, for each $\psi\colon Z \to X$ and each factorization $p\psi = uv'$, there exists a unique $\psi'\colon Z \to Y$ with $p\psi' = v'$ and $\psi = \phi\psi'$". Such a ϕ is called *cartesian over u* (with codomain X).

(ii) A *splitting of the fibration p* is a choice, for each pair $(X, u\colon J \to pX)$, of such a cartesian map $\phi_{(u,X)}\colon Y \to X$, satisfying the functoriality conditions of (9.2). A *split fibration* is a fibration equipped with a splitting.

(iii) A *cartesian functor* from p to the fibration $p'\colon \mathbf{F}' \to \mathbf{B}$ is a functor $f\colon \mathbf{F} \to \mathbf{F}'$ such that $p'f = p$, and for each cartesian map $\phi \in \mathbf{F}$, $f(\phi)$ is cartesian.

(iv) If p and p' are equipped with splittings, f is a *split functor* if, for each $(X, u\colon J \to p(X))$, $f(\phi_{(u,X)}) = \phi'_{(u,f(X))}$.

Thus, for each category $\mathbf{B}$, we can define the categories **Fib(B)** and **Split(B)** of fibrations and split fibrations over $\mathbf{B}$, and the underlying functor $\mathfrak{U} = \mathfrak{U}_{\mathbf{B}}$: **Split(B)** $\to$ **Fib(B)** which forgets about the splittings. Actually we can also define natural transformations over $\mathbf{B}$ between cartesian or split functors, and thus enrich **Fib(B)** and **Split(B)** to get 2-categories, and $\mathfrak{U}_{\mathbf{B}}$ becomes a 2-functor. Although this enriched structure is mathematically very important, it has no influence in our discussion and we shall omit it, for the sake of simplicity.

(9.4) We can now summarize some major points of our analysis.

(9.4.1) A "category" over $\mathbf{B}$ must at least (see (4.2)) be a fibration over $\mathbf{B}$, an equality of objects for such a "category" being then a splitting of the fibration (we know such a splitting need neither exist nor be unique). A "functor" thus becomes a cartesian functor, and a "functor" respecting the equality of objects is just a split functor between split fibrations.

(9.4.2) In naive category theory although we talk about, and think about, a category **C**, in fact we use the whole (split) fibration $p_{\mathbf{C}}$.

(i) The properties of **C** preserved by equivalence of categories are exactly those which are independent of the equality of objects, i.e. the properties of $p_{\mathbf{C}}$ as a fibration.

(ii) The properties of **C** preserved only by isomorphisms depend on the equality of objects, i.e. are properties of $p_{\mathbf{C}}$ as a fibration equipped with a splitting. An example, of major importance, of such properties is: "**C** is small".

§10. "Categories" and fibrations. (10.1) There are of course finitely axiomatizable first order theories $\mathbf{T}_{\text{fib}}$ and $\mathbf{T}_{\text{split}}$ having as models the fibrations and the split fibrations. In naive category theory, when we speak of a category **C** we have seen that implicitly we assume a lot of relations between **C** and **Ens**. We have extracted one such relation, via the notion of families, and built it into the fibration $p_{\mathbf{C}}$.

(10.2) We claim that $p_{\mathbf{C}}$ is all the information we need, and that *naive category theory essentially speaks, in a first order language, and with a finite number of axioms, about fibrations and split fibrations.* More precisely:

(i) All important notions involving **C** and sets (e.g. $(X_i)_{i \in I}$ is a family of generators of **C**, $S(X)$ is the set of subobjects of an object X, etc. ...) can be expressed in the first order language of $\mathbf{T}_{\text{fib}}$ or $\mathbf{T}_{\text{split}}$.

(ii) All implicit or explicit assumptions made about **C** and its relations to sets, and also about sets themselves, amount to the adjunction of a finite number of axioms to $\mathbf{T}_{\text{fib}}$ or $\mathbf{T}_{\text{split}}$. A category **C** will satisfy these assumptions iff the fibration $p_{\mathbf{C}}$ satisfies these extra axioms.

(iii) All major results of naive category theory (theorems of Kan, adjoint functors, Giraud, etc....) can be proved, without any reference to sets, for arbitrary fibrations satisfying axioms of type (ii).

(iv) All we have said about categories extends, with obvious modifications, to functors by using finitely axiomatizable extensions of $\mathbf{T}_{\text{fib}}$ and $\mathbf{T}_{\text{split}}$ which have as models the cartesian functors and the split functors.

(10.3) These are very strong claims. The only way to support them is no longer in the domain of foundations but of mathematics, namely to realize the program which they contain. This work is in progress, and the results already obtained are enough justification; cf. [Bé.1–4], [Cel] and [Mo].

(10.4) One might think that, since the notion of "category" coincides with the notion of fibration, the theory of "categories" reduces to the theory of fibered categories (with perhaps some considerations about splittings). We will show that this is not quite true.

(10.4.1) As fibered categories may not be very familiar, we will use an analogy, which is not superficial, and compare category theory and categorical logic. The objects of interest in both cases are the categories. However there is a major difference of attitude towards categories. Category theory is neutral and static, whereas categorical logic is partial and dynamic. It is based on the conviction that any category should be considered as some kind of approximation of "the" category of sets, and its driving force will tend to make such approximations better and better. This difference of attitude has important mathematical implications.

(i) *In the methods*: diagrams, infinite limits, etc.... in category theory, internal languages, tools from logic and proof theory, finitistic and constructive methods, in categorical logic (even if in examples, or in models the whole strength of ZF is used, because ZF then appears as the *metatheory*).

(ii) *In the choice of "interesting" assumptions*: e.g. starting with the notion of regular category, in view of the numerous important examples, in category theory one will strengthen the axioms by looking at abelian categories or perhaps at algebraic, or locally finitely presented categories. By the standards of categorical logic this is a deviation from the main stream. There are such standards, and such a stream—and natural axioms would be *motivated by sets*, e.g. the subobjects of any object form a distributive lattice, which would immediately rule out such important examples of categories.

(10.4.2) In much the same way, "categories" are fibrations $p: \mathbf{F} \to \mathbf{B}$, but considered from a specific point of view. We again pretend that $\mathbf{B}$ is some category of "sets", and moreover that p behaves with respect to $\mathbf{B}$ as a fibration of the form $p_{\mathbf{C}}$ with respect to **Ens**. And this, even when $\mathbf{B}$ is actually **Ens**, will again quickly rule out very important sorts of fibrations which have lots of properties in the general context of fibrations. Here is an example. For any category $\mathbf{X}$, we define the constant fibration (over $\mathbf{B}$) with fiber $\mathbf{X}$ to be the projection $p_1: \mathbf{B} \times \mathbf{X} \to \mathbf{B}$. As a fibration it is very simple, important, has a canonical splitting, and if you assume strong things about $\mathbf{X}$ (limits, exactness properties, etc....) they will give strong properties to the fibration p_1. However, even for $\mathbf{B} = \mathbf{Ens}$, we "have the feeling" that it is a very poor approximation to a fibration of type $p_{\mathbf{C}}$, where the fibers are the categories $\mathbf{C}^I$, and therefore not constant, except in the trivial case where $\mathbf{C} = \mathbf{1}$. Well, this feeling is justified: if we think of p_1 as a "category", the isomorphisms and equality (of parallel pairs) will in general not be definable (compare with (8.3)). More precisely, and this is now a *theorem* (easy):

(10.4.3) *Let* $\mathbf{B}$ *have at least one object* (*that is all we need here about our "sets"!*). *Then*:

(i) *Isomorphisms are definable for* p_1 *iff* $\mathbf{X}$ *is a groupoid* (*i.e. all maps are isomorphisms*).

(ii) *Equality is definable for* p_1 *iff* $\mathbf{X}$ *is a preordered class* (*i.e. there is at most one map between any two objects of* $\mathbf{X}$).

Notice that (i) and (ii) already tell us that *both notions are definable iff* $\mathbf{X}$ *is equivalent to a discrete category*. We can do much better.

(iii) p_1 *is locally small iff* $\mathbf{X}$ *is equivalent to* $\mathbf{1}$.

And since p_1 has a canonical splitting, i.e. an equality of objects, it makes sense (cf. (8.9.1)) to ask whether it is definable, and we get:

(iv) *Equality of objects is definable for* p_1 *iff* $\mathbf{X}$ *is isomorphic to* $\mathbf{1}$.

Thus, on the basis of very mild, and very natural, assumptions on the "category", p_1 will have to be rejected, whereas as a fibration, provided we make suitable assumptions on $\mathbf{X}$ and $\mathbf{B}$, it will satisfy the most drastic requirements which are usually made about fibrations.

(10.5) This example tells us, moreover, that definability, apart from being of major importance, will turn out to be a very strong notion.

§11. Forgetting about foundations. Most mathematicians and many logicians are no longer interested in foundations and tend to react negatively to such matters, especially when they involve questioning the role of set theory as expressed in ZF. Thus, although they do provide nice simple and powerful foundations for most of mathematics (excepting category theory), it was perhaps a psychological mistake to try to "sell" elementary toposes mainly on this basis as was done in the earlier stages of topos theory, and this may have hindered their spreading. Many mathematicians who could have found in topos theory a coherent and adequate complex of ideas, concepts, and results available as an effective tool in their work, just did not want to hear about foundations.

Hence, in spite of the title, I shall *not* try to make an issue of the fact that fibered categories, with the point of view described in (10.4.2), provide adequate foundations for category theory, even if I am convinced of this fact. However, I would like to convince you that they are worth looking at, for reasons which are both aesthetical or conceptual, and *mathematical.* The two aspects are of course closely related.

(11.1) Even if—all the more if—one believes that sets should never be questioned, their role in category theory should be understood. Our analysis shows that this role (not some specific notion of sets) is essential, and at the same time completely obscure, implicit and very deeply hidden. The notion of fibration, and the derived notions, such as definability, provide a simple and well-adapted frame where this role can be clarified both conceptually and syntactically.

(11.2) If one is ready to accept that, independently of all foundational questions, topos theory is, and will become more and more, an important domain of mathematics, in the mere development of this domain there will occur (and has already occurred) the necessity of having some sort of "naive category theory with sets replaced by the objects of a topos". The possibility of having such a theory, and a description of its main features, can come only after the role of sets in naive category theory has been clarified.

(11.3) Again, without questioning ZF, we can accept that naive category theory, since it talks about such notions as "all sets" or "all groups" etc. . . . , is not part of ZF, but belongs to the *model theory* of ZF. But then, without shattering any idol, one may ask some questions: What happens to naive category theory if we change the model of ZF? What reasonable relations should exist between two models of ZF in order to be able to compare the naive category theories associated to these models? What kind of categorical properties shall be preserved? How shall the other properties be affected by this change? And so on.

(11.4) Answers to such questions will, in our context, need a careful and thorough study of the change of base for fibrations, which thus has to be a major chapter in the study of fibrations.

The importance of the change of base was well known in the general theory of fibrations, but here again we have a "point of view" (10.4.2) about fibrations; thus our investigation about the change of base shall not be neutral, and we shall be "more interested" in some properties of the change of base (preservation of definability, etc. . . .).

The relevance of this point of view is that it clarifies, and thus makes easy to work with, many fundamental notions. For example: why are the geometrical morphisms the appropriate notion of morphisms between toposes? (Just looking at the axioms of topos theory, one would rather expect logical morphisms). Of course, from geometry, we know they are important, and then from actual work in toposes we know that we can prove many things about them. But why is that so? The answer, in terms of fibrations, is quite simple: As change of bases they preserve some fundamental properties of "categories", i.e. fibrations, and this characterizes them uniquely, whether the bases are toposes, or not.

(11.5) The fact that our "categories", "functors" and "sets" are finitely axiomatizable (by this we mean not only fibrations and cartesian functors, but also any further assumptions we need to make about them) permits us to internalize the whole theory, i.e. to take models in a topos (again, for most purposes, toposes can be weakened).

This has a mathematical content. Special cases of such models have been introduced, as locally fibered categories, in [SGA].

(11.6) But now I should be ready to face the following type of objections: "If you agree to forget about foundations, which anyway I don't care about, what all this boils down to is that you remark the well-known and trivial fact that, for any category **C**, the functor $p_{\mathbf{C}}$ is a fibration equipped with a canonical splitting, and emphasize this fact. Then you tell us that fibrations and split fibrations are important, but we already knew this from Grothendieck, Giraud and many others. Why all the fuss?" Or again: "This finitely axiomatizable system is very poor, and not much serious mathematics can probably arise out of it." To these objections, I can give the following answers:

(i) All of elementary topos theory is contained in the "remark", *even more* "well-known and trivial", that the category of sets admits finite limits and power sets. But what was added, and this was a very deep insight of Lawvere, was the emphasis on this remark, the point of view that toposes were generalized categories of sets, and the claim that most of mathematics could be carried inside toposes. (And again, such a claim can only be "proved" by actual work with toposes (cf. (0.9)).)

(ii) Even if I forget about foundations, I am *not* willing to forget about the point of view, and the analysis; and these contain much more than mere fibrations (definability, smallness and local smallness, etc...., and in fact all the notions of naive category theory).

(iii) Although, at the beginning, nothing compels us to assume, for fibrations over **B**, that **B** is an elementary topos, our formal language is rich enough to formulate such an assumption. (We even know that, in order to get deeper properties of such fibrations, we will eventually have to do so.) Thus, the language of fibrations is richer than the language of elementary toposes.

§12. Appendix. Fibrations and indexed categories. (12.1) Fibrations were introduced in [Gr] to axiomatize in terms of descent all the "gluing-processes". In this paper some of their very basic properties were studied by Grothendieck, and he gave in great detail the following construction:

If $p: \mathbf{F} \to \mathbf{B}$ is a fibration, using the axiom of choice for classes, one can choose for each pair $(X \in \mathbf{F}, u: J \to I = p(X) \in \mathbf{B})$ a map $\phi_{(u,X)}: Y = u^*(X) \to X$ satisfying the universal property (9.3)(i). Such a noncanonical choice then determines, for each $u: J \to I$, a unique functor $u^*: p^{-1}(I) \to p^{-1}(J)$.

Since the $\phi_{(u,X)}$ fail to satisfy (9.3)(ii), for $v: K \to J$ we have $(uv)^* = v^*u^*$. However (9.3)(i) permits us to define a unique isomorphism $c_{u,v}: v^*u^* \rightleftarrows (uv)^*$ satisfying a precisely stated property which we do not bother to write. And again the uniqueness of the $c_{u,v}$ forces them to satisfy some very strong coherence conditions. The data $(p^{-1}(I), u^*, c_{u,v})$ satisfying these coherence conditions is called a *pseudofunctor* from $\mathbf{B}^{op}$ to the category **Cat** of categories. The choice of the $\phi_{(u,X)}$ is a *cleavage* of p.

This means, in our simple-minded approach, that for any "category", one can choose, in a noncanonical way, a pseudoequality of objects which "behaves badly", and in general will not be definable (the coherence conditions mean that it cannot behave too badly, since it reflects some of the properties of equality). This can be "predicted" at the very unsophisticated level of (6.2): just choose for each representable class a representative of this class.

Conversely, for each pseudofunctor from $\mathbf{B}^{op}$ to **Cat**, Grothendieck constructed a cleaved fibration, and proved that the notions of pseudofunctor and cleaved fibrations are equivalent. He also showed how to define cartesian functors, not cleavage-preserving, between cleaved fibrations in terms of pseudofunctors. This again involves coherence conditions on some given natural isomorphisms. The proofs are long and tedious, but straightforward verifications, mostly left to the reader because they would add nothing to our understanding of fibrations, and moreover one is convinced from the beginning that the result has to be true.

We must recall that *in 1959, Grothendieck had already defined the pseudofunctors* (which he then called fibered categories [TD]), before we quote his conclusion, omitting only a few words where notations are involved.

> Bien entendu, il y a intérêt le plus souvent à raisonner directement sur des catégories fibrées sans utiliser des clivages explicites, ce qui dispense en particulier de faire appel, pour la notion simple de... foncteur cartésien, à une interprétation pesante comme ci-dessus. C'est pour éviter des lourdeurs insupportables, et pour obtenir des énoncés plus intrinsèques, que nous avons dû renoncer à partir de la notion de catégorie clivée..., qui passe au second rang au profit de celle de catégorie fibrée.
>
> Il est d'ailleurs probable que, contrairement à l'usage encore prépondérant maintenant, lié à d'anciennes habitudes de pensée, il finira par s'avérer plus commode dans les problèmes universels, de ne pas mettre l'accent sur *une* solution supposée choisie une fois pour toutes mais de mettre toutes les solutions sur un pied d'égalité.

Without twisting the ideas, this very far reaching conclusion, written in 1961, tells us that we should forget about equality of objects, at least in the following cases:

(i) All we can define is a noncanonical pseudoequality.
(ii) We study *all* "functors", hence this equality is not preserved (8.9.2).
(iii) We are interested in categorical properties of objects (8.2).

(12.2) We shall now describe successive "improvements" going from fibrations to the present version of indexed categories as can be found in [Par-Sch].

(12.2.1) The first step was just to revert to pseudofunctors, to rebaptize them *indexed categories*, and to forget completely the (cleaved) fibration determined by such a pseudofunctor.

This attitude is tenable, as long as you work with a single pseudofunctor, although many important questions are avoided such as what properties are intrinsic (i.e. independent of the cleavage), and how do you test such properties? It becomes much more uncomfortable if you want to study cartesian functors and properties of "all" fibrations, without even mentioning what a fibration is. (You can probably study one group defined by generators and relations, without mentioning or knowing what a group is; but try to define, just in terms of generators and relations, a homomorphism of groups, let alone to study the category of groups.) It is highly significant that Grothendieck defined pseudofunctors in [TD] but did not say, there, what their morphisms were. It was only after, and because, the intrinsic notion of fibered category was discovered, that these morphisms were given. Of course technically they could have been defined before, but their motivation, and their meaning, came from fibrations. In terms of pseudofunctors alone, the relevant notion of morphism would have been either more strict or more lax: The ϕ_f defined in [Gr] being either identities, or natural transformations (not isomorphisms) satisfying coherence conditions.

(12.2.2) The second step is based on the fact (cf. (12.1)) that a cleavage determines unique isomorphisms $c_{u,v}$. These are thought to be canonical. Hence the pseudofunctors are now assumed to take their values, not in **Cat**, but in the category of categories equipped with a class of isomorphisms, called *canonical*, and otherwise not specified (in the examples they will be either the identities or all isomorphisms).

I think there is a slight confusion here. There is nothing canonical about the $c_{u,v}$; they depend on the highly noncanonical choices involved in the cleavage. There is a major difference between "canonical" and "uniquely determined after some (noncanonical) choices have been made". A typical example, closely related to the situation of fibrations and cleavages, is the following. Let $F\colon \mathbf{C}^{\mathrm{op}} \to \mathbf{Ens}$ be a functor. The identity natural transformation $\mathrm{id}_F\colon F \to F$ is canonical. Suppose F is representable, and choose two representatives X and Y of F. By the Yoneda lemma, to id_F will correspond a uniquely defined map $f\colon X \to Y$, which has nothing canonical. Even if you force Y to be equal to X as objects, all automorphisms of X can be obtained as such f's. I apologize about such trivialities, but this tells us that "canonical" should have a meaning only when we have an equality of objects, and that the meaning should depend on this equality.

This confusion is carried a bit further. An indexed functor between indexed categories is a very complex notion, one of the ingredients of which is (cf. [Par-Sch, I(1.2)]) a family of functors F^I defined up to canonical isomorphisms. This definition does not make very much sense. Moreover, in the context of fibrations, if you have a cartesian functor, and if you choose to describe it in terms of cleavages, the u^*X, $c_{u,v}$, ϕ_f, etc. ... will depend on the choice of the cleavages. The only thing you can be sure of is that the functors F^I are perfectly and uniquely determined, independently of any cleavage. But now the complexity of the situation and the confusion about "canonical" have mathematical consequences: because the situation is too complex to be completely controlled the authors sometimes forget completely about the canonical isomorphisms, or sometimes pretend they work "up to" canonical isomorphisms. Thus some definitions are ambiguous, with many possible meanings left to the reader, *not equivalent*. And in different places the different meanings are used as if they were the same. But this has an influence on the results: some are true but have false proofs, some are false, and some do not have any meaning.

We shall give a few significant examples in (12.5), but more important to the spirit of this paper is the fact that, on the basis of our intuitive discussion of equality of objects, such a muddled situation was *bound* to arise: Instead of accepting the fact that there are "categories" without equality of objects, and learning how to work with them (which is very easy since categorical properties do not involve such an equality), the first step is to introduce an arbitrary, artificial pseudoequality which we know has to behave badly. The second step is to replace one such pseudoequality by many partially defined pseudoequalities, very loosely related. Of course, as stated by Grothendieck in his conclusion, the situation will immediately, and needlessly, become complicated when you start working with "functors" which do not preserve such pseudoequalities.

And pretending to work "up to canonical isomorphisms" will just add to the confusion, because in category theory you never work "up to" any kind of isomorphisms unless these are unique (e.g. for subobjects of an object), since this would amount, in particular, to "killing" all the "canonical" automorphisms (cf. (12.5)(iii)).

(12.3) We shall now assume that indexed categories have been corrected, and also that the knowledge and control of general coherence conditions has progressed since [Gr], so that most tedious verifications have become trivial, and under these assumptions we shall compare indexed categories and fibrations, both as a mathematical tool and as foundations.

(i) With indexed categories, although it is never done, one should worry about questions of the following type: What is intrinsic? What happens if the cleavage or the "canonical isomorphisms" are changed? On the other hand, when you work with fibrations everything is automatically intrinsic.

(ii) With fibrations the coherence conditions are replaced by unicity conditions. Thus you need not worry about them; they "take care of themselves".

(iii) An indexed category is a very complex notion involving "infinitely many" categories, functors, and natural transformations, whereas a fibration is just one functor. Moreover, this functor satisfies a very strong property, and this is not a handicap but a help in the study of the functor.

(iv) By looking only at the fibers, and refusing to consider the total fibration, the notion of indexed category makes obscure, or even almost impossible to formulate, some obvious and important facts, e.g.: If $p\colon \mathbf{F} \to \mathbf{B}$ and $q\colon \mathbf{C} \to \mathbf{F}$ are fibrations, so is the composite pq. Now if you try to replace p by an indexed category, you will have only the fibers of p, not the category $\mathbf{F}$. And hence you cannot even define q. Of course, if you are willing to pay the price, there is an "indexed category version" of this trivial fact, but it involves "lax Kan extensions of pseudofunctors", and the mere statement, let alone the proof, would take many pages.

(v) As we already mentioned, an indexed category is just a presentation of a fibered category. And, just as in group theory, you should try to work intrinsically, but nothing prevents you, when you study some specific fibrations, from using such presentations, provided you remember that the important thing is the fibration. But in all "versions" of indexed categories you are not even supposed to know that "something" is presented, let alone "what" is presented.

Of course all this was said, more briefly and beautifully, in Grothendieck's conclusion in 1961. Moreover it was emphasized in great detail, with many mathematical examples, in my lectures in 1974, where Paré and Schumacher were present.

(12.4) We now compare the two notions from the point of view of foundations.

(i) There is a first-order, finitely axiomatizable theory of fibrations which is perfectly satisfactory (cf. (10.2)) and fits nicely in the general process of finite axiomatization initiated by Lawvere for the notion of "sets" with the elementary toposes. But there is no theory whatsoever of indexed categories, finitely or infinitely axiomatizable. The notion of indexed category belongs to model theory, i.e. to naive category "theory", and we end up where we started, without any foundation. (The analogy with groups still works, a presentation belongs to model theory of groups, and there is no first-order theory of presentations.) This has very strong mathematical implications (cf. (11.5) and (ii) below).

(ii) We can sharpen (i). An indexed category is a presentation of a fibered category equipped with a cleavage. If we use the axiom of choice, every fibered category admits such a cleavage. But without AC cleaved fibrations are much stronger than fibrations. There is again a first-order, finitely axiomatizable theory. $\mathbf{T}_{cl}$ of cleaved fibrations, of which $\mathbf{T}_{split}$ is a strengthening. But now $\mathbf{T}_{cl}$ and $\mathbf{T}_{split}$ are $\varprojlim$ theories (called in [SGA] definable by finite $\varprojlim$'s). This means that we can internalize the notions of cleaved or split fibrations, i.e. take models of $\mathbf{T}_{cl}$ and $\mathbf{T}_{split}$, in any category with finite $\varprojlim$'s (actually only pull-backs are needed).

But there is no way to define an internal indexed category. All you will try to do will automatically involve the whole internal cleaved fibration, i.e., as "predicted" in (3.6), a gluing together of all the fibers in a global structure. This is true even in the case of discrete indexed categories, which correspond to discrete fibrations. Again the only way to internalize the notion is to glue together all the fibers. It is to me very surprising that the advocates of indexed categories have used internal diagrams intensively in topos theory [Joh, II(2.14)], have been aware of the internalization process involved there, and of the necessity for this process to have a first order theory [Joh, II(2.13) and remarks before (2.14)], and yet have not realized that a generalization of this process implied a global definition of "indexed categories", and a first order theory of type $\mathbf{T}_{fib}$, $\mathbf{T}_{cl}$ or $\mathbf{T}_{split}$, and thus a definition of "indexed categories" as fibrations.

In our frame, by adding one axiom to $\mathbf{T}_{cl}$ we can define the theory $\mathbf{T}_{disc}$ having as models discrete fibrations. It is again a $\varprojlim$ theory, and internal diagrams are models of this theory in any category with finite $\varprojlim$'s. But now, if $\mathbf{C}$ is a category with finite $\varprojlim$'s and $\mathbf{T}$ is any $\varprojlim$ theory, the properties of models M of $\mathbf{T}$ in $\mathbf{C}$ involving only diagrams constructed from M by taking pull-backs, finite products, and repeating the process a finite number of times, can be proved by just checking them for $\mathbf{C} = \mathbf{Ens}$, where in general they are trivial (a fact which has been known for 25 years). If you are more "logically" minded, this can be stated as a metatheorem. Taking $\mathbf{T} = \mathbf{T}_{disc}$, this makes pages of diagrams about internal categories and diagrams explicitly displayed in some papers perfectly useless. For $\mathbf{T} = \mathbf{T}_{cl}$, this tells us that, whereas indexed categories cannot be internalized as such, cleaved categories can, at no cost at all.

(12.5) *A few basic typical mistakes.* In this subsection the terminology, notations and references are [Par-Sch].

(i) The very first definitions of indexed categories and functors do not make sense; specifically I(1.1)(ii), α^* defined up to canonical isomorphism, and I(1.2)(i), F^I defined up to canonical isomorphism. Notice that all the paper is built on these definitions.

We will pretend in what follows that this is "harmless".

(ii) It is not assumed in I(1.2) that the F^I preserve canonical isomorphisms. Hence in II(1.3) it does not follow from $\underline{A} \simeq \underline{B}$ and $\underline{B} \simeq \underline{C}$ that $\underline{A} \simeq \underline{C}$! Supposing now that this is a "minor" omission and that canonical equivalence is made transitive, again from (1.3) it follows that two categories [$\mathbf{C}$] and [$\mathbf{D}$] are canonically equivalent if $\mathbf{C}$ and $\mathbf{D}$ are isomorphic. In particular for $\underline{S} = \underline{\text{Set}}$ this tells us that "canonical equivalence" has nothing to do with equivalence of categories. Notice that in I(2.1), which is the fundamental example, no mention is made of the so-called canonical isomorphisms. You "may" think they are the identities, but then canonical equivalence of categories is just isomorphism. However, if $\underline{\text{Set}}$ is considered as a topos you "may" no longer take the identities (cf. I(2.2)).

(iii) In II(2.1) the "discrete category of objects" is nothing but discrete. In particular, for $A = S$, since all isomorphisms in the $\underline{S}^I$ are canonical, this indexed category $\underline{D}$ is defined by $\underline{D}^I =$ the groupoid of isomorphisms of $\underline{S}^I$. This would amount to saying that the category having as objects the sets and as maps the bijections is discrete (or equivalent to a discrete category), thus killing all automorphisms. Notice that all smallness considerations, which are the very core of all the subject, are based in [Par-Sch] on such "discrete" categories of objects, which do not make any sense.

(iv) As already mentioned, the category of indexed categories need not have pull-backs, or some pull-backs may exist but not be computed fiberwise. However the pull-backs involved in the construction of (F, G) in I(2.10) exist and are computed fiberwise, but *this has to be proved* (I am not even sure that it is true if you throw in arbitrary canonical isomorphisms; or that it even makes sense (cf. (2))).

(v) The process of "variable definitions" we mentioned earlier is carried to its ultimate limit with functor categories—In III(1.1) no mention at all of the canonical isomorphisms of $\underline{A}^{\underline{B}}$ is made, not even a hint at what they should be. Yet the authors manage to give a (sketch of) "proof", again without any mention of canonical isomorphisms, of proposition (1.1.1). Well, this proposition is false; it would imply that any fibration has a splitting.

Moreover the definition of $\underline{A}^{\underline{B}}$ uses the $\underline{A}^I$ of I(2.8), and thus depends on the fact that $\underline{S}$ has finite products. This is not satisfactory in view of our (6.5.2). Functor categories should be defined without any assumption on $\underline{S}$. (A well-known special case of this remark goes as follows: looking only at discrete indexed categories we get, for an arbitrary $\underline{S}$, the topos $\hat{\underline{S}}$ of presheaves on $\underline{S}$. Hence for $F, G \in \hat{\underline{S}}$ we can define $F^G \in \hat{\underline{S}}$. Even if it happens that $\underline{S}$ has products, one would not make the definition of F^G to depend on such products.) A correct definition of functor categories is to be found in [Gi, I(3.3)].

(vi) I could give many more examples of such ambiguous or false statements. I chose (i)–(v) because they are the keystones of all the subject, since they involve the very definition of indexed categories and functors, the notion of equivalence, all smallness conditions, and functor categories. Of course, some of them could be corrected, but at the cost of a horrible complexity, and still leaving us with the fact that an indexed category is just a presentation of some "global entity" never to be mentioned or acknowledged.

Looking at things from a more general point of view, in many domains of mathematics we have important notions of "families", e.g. vector bundles, homotopies, families of analytic manifolds, etc...., from which we can "extract" individual elements (vector spaces, continuous maps, analytic manifolds, etc....) and study relations between these elements, but the family is the global entity. Why should "families of categories indexed by a category" behave differently? Why should the emphasis be made on the fibers, let alone the fiber over the terminal object of $\underset{\sim}{S}$, the existence of which is not even needed in many questions?

(12.6) *Some remarks about chronology.* From the preface of [Joh-Par] we extract the following quotation (the numbers (1) to (6) have been added to make our comments more precise).

> However in order to develop category theory itself this way, it was realized that one needed to develop a notion of "family of objects".... This realization was first made by Lawvere himself who mentioned it in his lectures at Dalhousie University in 1970.(1) In 1972–73 he developed, in unpublished notes written in Perugia, a detailed theory of families for complete categories with small homs.
>
> Independently J. Penon was developing his theory of locally internal categories...from the point of view of enriched categories.(2) Then(3) J. Bénabou and J. Celeyrette, knowing of Penon's work(4) but not of Lawvere's, developed their theory of families using fibrations. Their results were presented by Bénabou...in the summer of 1974.
>
> In 1973–74 R. Paré and D. Schumacher felt the need(5) for a theory of indexed categories...their own development of the theory was well advanced(6) when they received his Perugia notes and then, a little later, attended Bénabou's lectures in Montreal.

(1) I started, not only "realizing" but working on fibrations as foundations for category theory in 1970: Celeyrette's thesis was completed in 1974, a "thèse de doctorat d'État" in France takes at least 4 years. At the same period (1970), and independently, Lawvere started working on indexed categories as a tool to do "category theory over a base topos". I still do not know the content of his Perugia papers, but he is too profound a mathematician to have included there (at least without precautions) the so-called canonical isomorphisms, or to have believed that indexed categories, i.e. pseudofunctors, could provide foundations for category theory. At that moment, he was probably not preoccupied about foundations of category theory, because he was involved in the process, of major importance, of

developing topos theory. It so happens that fibered categories provide both foundations and a better tool to do "naive category theory without sets".

(2) Penon's approach is completely misleading: Local smallness is not an extra structure but a property. The enriched structure is completely and uniquely (up to isomorphism) determined by the fibration. It was started as such in Celeyrette's thesis, and in my lectures in 1974. Moreover, it is one of the strongest properties that a fibration can have, and it implies many important consequences on the fibration (e.g. (10.4.3)(ii)). This could be predicted from our simple-minded analysis (cf. (6.3)). Yet even in 1978, Paré and Schumacher were not aware of this fact, they just say that local smallness is "slightly better than having a category enriched over $\underline{S}$" and should be viewed as "a property rather than an extra structure"!

Apart from the major difference between Penon's approach and ours, (3) and (4) are utterly false. Penon's work was done in 1974. By that time Celeyrette's thesis was completed. Local smallness appears very early in this thesis, and all the major results depend on this notion, which was defined in 1972, long before Penon's work. Thus we could hardly have "known about" it!

(5), (6) When, in 1973–74, Paré and Schumacher started to "feel the need", Ceylerette's thesis was already very much advanced, and my own work even more. Of course, they could not have known that. However, my lectures of 1974, at which they were present, covered almost all of what they published in 1978.

I did not publish these lectures because I had the justified feeling that something vital was missing in order to achieve the program I had decided on in 1970. It was the notion of *definability*, which of course does not appear in [Par-Sch]. I found it in 1977, but needed some time, in view of its importance, to "test" it in many significant situations in order to convince myself that it was a keystone of the subject, as much as the comprehension scheme in set theory.

Short annotated glossary. Let $p: \mathbf{C} \to \mathbf{B}$ be a fibration, thought of as a "category".

(1) A *class of objects of the "category"* p is a class $\mathscr{C}$ of objects of $\mathbf{C}$ such that if $X \in \mathscr{C}$ and $f: Y \to X$ is cartesian, then $Y \in \mathscr{C}$. Given $\mathscr{C}$, if X is an object of $p^{-1}(I)$ we can define a sieve $\mathscr{C}_X$ of $\mathbf{B}/I$ by $u: J \to I \in \mathscr{C}_X$ iff there exists $f: Y \to X$, cartesian over u, such that $Y \in \mathscr{C}$. We say that $\mathscr{C}$ is a *definable* class of objects of p iff all the sieves $\mathscr{C}_X$ are representable. If X is viewed as a "family" $(X_i)_{i \in I}$, a representative of $\mathscr{C}_X$ is a subobject J of I, which must be "thought of" as $J = \{i \in I \mid X_i \in \mathscr{C}\}$. (Our sets can "detect" among "the X_i's" those which "belong to $\mathscr{C}$".) This definability extends to various notions. The most important are:

(2) *Definability of equality* (cf. (8.1)): If $f, f': Y \rightrightarrows X \in p^{-1}(I)$, call $\mathscr{E}(f, f')$ the sieve of $\mathbf{B}/I$ defined by $u: J \to I \in \mathscr{E}(f, f')$ iff there exists $g: Z \to Y$ cartesian over u such that $fg = f'g$. If all these sieves are representable, we say that equality (of parallel pairs) is definable. A subobject J of I representing $\mathscr{E}(f, f')$ is thought of as $J = \{i \in I \mid f_i = f'_i\}$.

(3) *Definability of isomorphisms* (cf. (8.2)): If $f: Y \to X \in p^{-1}(I)$, call $\mathscr{Iso}(f)$ the sieve of $\mathbf{B}/I$ defined by $u: J \to I \in \mathscr{Iso}(f)$ iff there exists $g: Z \to Y$ cartesian over u such that fg is cartesian. If all these sieves are representable, we say that isomorphisms are definable. Again a representative of Iso(f) "is" the "set" $J = \{i \in I \mid f_i$ is an isomorphism$\}$.

Like all representability conditions, (2) *and* (3) *are very strong* (cf. (10.4.3)). Naively they tell us that our "sets can detect the property" of being an isomorphism, or, for parallel pairs, of being equal.

(4) For $X, Y \in p^{-1}(I)$ we can define (cf. [Gi]) a functor $\mathcal{H}om_I(X, Y)$ from $(\mathbf{B}/I)^{\text{op}}$ to **Ens**. If all these functors are representable, the "category" is called *locally small*. This is a very strong property (cf. (10.4.3)(ii), which could be "predicted" from (6.3)).

(5) The "category" $p(\mathbf{B})$ which plays the role of "sets" is the *codomain functor*: $\mathbf{B}^2 \to \mathbf{B}$. In order for it to be a "category" at all (cf. (6.5.5)), i.e. a fibration, **B** has to have pull-backs. But as soon as this "category" exists, it has wonderful (!) properties, namely all finite $\underleftarrow{\lim}$'s and all internal "infinite" sums, which moreover are "disjoint" and "universal" (even if **B** has neither initial nor terminal object, and no finite sums!). This calls for a few comments: in naive category theory, when we say that a category **C** *has infinite sums*, it is in fact a statement about the fibration $p_{\mathbf{C}}$ (cf. (3.5)), i.e. it involves some kind of "comparison" between **C** and **Ens**. In particular when we say that **Ens** has infinite sums we are "comparing **Ens** with itself"! It is then a tautology that any category **B** "compared with itself" has "infinite" sums. One of the by-products of fibrations, with this point of view, is to clarify what are the "real", i.e. not tautological, properties of **Ens**, which via $p(\mathbf{B})$ will give rise to significant assumptions of type (4.3) about **B**. For example, *in general*, $p(\mathbf{B})$ *will not have definable equality or isomorphisms, and will not be locally small.* These are nontautological properties that **Ens** (or any topos) has. And for deeper considerations about fibrations over **B**, we will have to assume them for $p(\mathbf{B})$.

(6) Assuming that **B** has pull-backs, a category object $\underline{C}$ of **B** defines a *functor* (not pseudo) $\text{Hom}(-, \underline{C})$: $\mathbf{B}^{\text{op}} \to \mathbf{Cat}$ and hence a split fibration $p_{\underline{C}}$.

We say that $p: \mathbf{C} \to \mathbf{B}$ is a *small "category"* if it is equipped with an isomorphism with a fibration of form $p_{\underline{C}}$. Notice that this implies that p is equipped with a splitting (equality of objects) (cf. (9.4.2)(ii)) and that this *is not* a categorical notion.

(7) *Equality of objects revisited.* In many discussions on the subject there seems to have been some misunderstanding, along the following lines. In the very notion of a category, in order to be able to compose two arrows f and g you have to know that the objects domain(f) and codomain(g) are equal. Thus you cannot dispense with equality of objects, unless you do "something different from category theory". This is quite true, I know, everybody knows this equality of objects, but our "sets" need not "know" it or be able to detect it (cf. (1), (2) and (3)) with their internal properties or logic. As pointed out in (8.9.3)(ii), even in the restricted case of sets and classes, the *external equality* of two sets $S = T$ need not be definable in the model $\mathcal{M}$, unless you assume $S \equiv T$ if $S = T$—which I do not.

What I mean by a category with equality of objects, is a category where this equality can be "talked about" with my sets.

After all, *most of naive category theory consists in talking about properties of categories using sets as a language.* If an ordinary category is, say, not locally small, you cannot use sets to "talk about $\text{Hom}(X, Y)$", and you will correctly say that the category does not have Hom-sets. In exactly the same way, I say that a category does not have equality of objects if I cannot talk, within my sets, of $X = Y$. This equality then belongs to model theory, metamathematics, etc. .... It is this type of equality which is needed in order to be able to compose maps, not the "internal" version, which need neither exist nor be unique.

The "fibered translation" of this difference is that, for a fibration p: $\mathbf{C} \to \mathbf{B}$, $\mathbf{C}$ and $\mathbf{B}$ are "honest to god" categories, for which there is, as usual, an equality of objects, but this need not be represented in $\mathbf{B}$.

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