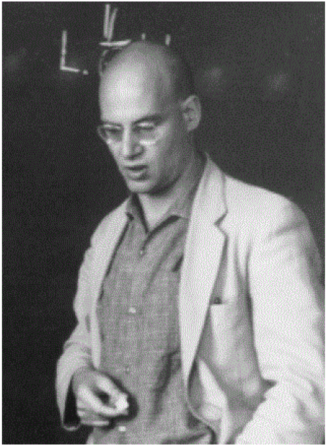

photo ca. 1965

The principal force in all my work has been the quest for the general. I prefer to accent "unity." But for me these are two aspects of one quest. Unity represents the profound aspect, and generality the superficial. – Grothendieck 1985

Grothendieck never cared about large sets – pro or con.

As Weil advised in *Foundations of Algebraic Geometry*, Grothendieck did not begin his innovations with perfect rigor.

Introduced them to seminars in 1958. Bourbaki, Chevalley.

Through the 1960s, seeking proofs, he needed large structures.

Posited "the notion of a Universe, a set 'large enough' that the habitual operations of set theory do not go outside it."

Two slogans for logicians:

Topos cohomology needs no replacement.

We need very little comprehension for the large structures of cohomology.

Founding EGA and SGA essentially unchanged, except a few passages on set theory.

Will argue that this, which amounts to cut elimination, rather than limited technical theorems on sites, is the true sense of saying the large structures can *always* be eliminated in favor of small.

All of this will be at the consistency strength of Finite Order Arithmetic = Simple Theory of Types with Infinity.

Begin with MC, for MacLane set theory including choice.

ZFC without foundation or replacement but with the separation axiom scheme for bounded (Δ_0) quantifiers.

Equivalently ETCS = Lawvere's elementary theory of the category of sets.

For non-specialists: this is the strongest theory commonly called "arithmetic."

And the weakest theory commonly called "set theory."

I assume specialists are familiar with the issues.

Standard textbook treatments of category theory typically fall into this set theory naturally.

Category theory is "typed," and often constructive or predicative without trying.

Accounts of category theory in a topos, as by Johnstone, entirely suit this set theory.

E.g. all forms of Yoneda lemma are provable here.

Each set-valued funtor on a small category is snall.

The presheaf category on a small category is a definable class.

Every presheaf on a small category is a (definable) colimit of representable presheaves.

Grothendieck used replacement to prove cohomology groups are defined.

For any small Grothendieck site, and sheaf of rings, every sheaf of modules embeds in an *injective*.

Beautiful proof in Grothendieck 1957, before sites were invented!

Two issues for MC:

To avoid countable replacement, work with the single set $R^{\mathbb{N}}$ of sequences on a ring R, rather than iterate steps on R.

Lift to toposes, without using replacement, by the 1974 Barr covering theorem.

Topos cohomology per se is a first order theory with trivial consistency strength:

An elementary (Lawvere-Tierney) topos with a ring whose modules have enough injectives.

In hindsight, Grothendieck used universes to connect toposes to arithmetic, to prove enough injectives, and to form categories of toposes etc.

Lighter weight apparatus, really adding only infinity, in fact suffices for all that.

Mac Lane type theory

Take the sets of MC as one type and add classes of sets as a higher type and collections of classes as another.

Indicate sets by italics x, A, classes by calligraphic $\mathcal{A}, \mathcal{B}$, and collections by fraktur $\mathfrak{A}, \mathfrak{B}$.

$A \in_1 \mathcal{A}$ for set A is in class $\mathcal{A}$

$\mathcal{A} \in_2 \mathfrak{B}$ for class $\mathcal{A}$ is in collection $\mathfrak{B}$.

The point is: classes and collections have only *set theoretic comprehension*.

Formulas defining classes and collections only quantify over sets.

Class inclusion is set theoretic:

$$\mathcal{A} \subseteq \mathcal{B} \;\leftrightarrow\; \forall x\,(x \in_1 \mathcal{A} \to x \in_1 \mathcal{B})$$

Collection inclusion is well defined but not set theoretic

$$\mathfrak{A} \subseteq \mathfrak{B} \;\leftrightarrow\; \forall \mathcal{X}\,(\mathcal{X} \in_2 \mathfrak{A} \to \mathcal{X} \in_2 \mathfrak{B})$$

For example, the collection $\mathcal{B}^{\mathcal{A}}$ of all functions $\mathcal{F}:\mathcal{A}\to\mathcal{B}$ between classes $\mathcal{A},\mathcal{B}$:

$$\mathcal{B}^{\mathcal{A}} = \{\mathcal{F} \mid \mathcal{F} \subset \mathcal{A}\times\mathcal{B} \ \& \ (\forall x \in_1 \mathcal{A})(\exists! y \in_1 \mathcal{B}) \ \langle x, y\rangle \in_1 \mathcal{F}\}$$

Quantifies over elements x, y of $\mathcal{A},\mathcal{B}$.

It uses class inclusion.

Another example, the class of all small categories:

$$\{\langle C_0, C_1, d_0, d_1, m\rangle \mid Cat(C_0, C_1, , d_0, d_1, m)\}$$

Here *Cat* is a formula about sets C_0, C_1, d_0, d_1, m and the set which is the 5-tuple.

There is even a collection of all class-sized categories:

$$\{\langle \mathcal{C}_0, \mathcal{C}_1, \mathcal{D}_0, \mathcal{D}_1, \mathcal{M}\rangle \mid \mathit{Cat}(\mathcal{C}_0, \mathcal{C}_1, \mathcal{D}_0, \mathcal{D}_1, \mathcal{M})\}$$

Deals with classes $\mathcal{C}_0, \mathcal{C}_1, \mathcal{D}_0, \mathcal{D}_1, \mathcal{M}$.

Quantifies over their elements.

We take tupling as an operation on classes.

So there is a (collection) category of all class-sized categories.

This type theory is adapted from Takeuti *Proof Theory*.

Such stricter-than-predicative comprehension is conservative over the base type, by cut elimination.

Michael Rathjen points out that is not cut-elimination in the strict sense, and the more direct comparison would be cut-elimination in showing GB conservative over ZF.

The identity axiom connects classes to sets:

$$\forall \mathcal{A} \forall x \forall y\, (\, (x = y \ \& \ x \in_1 \mathcal{A}) \rightarrow y \in_1 \mathcal{A}\,)$$

MTT has no identity relation for classes or collections.

So large structure theorems must specify the structures directly, not infer identity by extensionality.

This is normal for large structure proofs in EGA and SGA.

E.g. Grothendieck toposes are definable in MTT but the definition quantifies over functors of class type.

A category is a Grothendieck topos if there exists a functor equivalence between it and some sheaf topos.

MTT cannot prove there is a collection of all Grothendieck toposes.

It proves there is a collection $\mathfrak{Top}_0$ of all sheaf toposes – with specified sites – and thus all Grothendieck toposes up to equivalence.

A *locally smal category* has a class of objects, and any two objects have a set of arrows between them.

The standard theorems on the (collection) category $\mathfrak{Top}$ of all sheaf toposes follow in MTT.

They make elementary use of classes, and quantify over sheaves and other sets.

Theorem (Comparison lemma)

Let a localy small site $\langle \mathcal{C}', J' \rangle$ have a full and faithful functor $u : \mathcal{C} \to \mathcal{C}'$ from a small category $\mathcal{C}$ where every object of $\mathcal{C}'$ has at least one J'-cover by objects $u(A)$ for objects A of $\mathcal{C}$.

Then J' induces a topology J on $\mathcal{C}$ making $\widetilde{\mathcal{C}}_J$ and $\widetilde{\mathcal{C}'}_{J'}$ equivalent categories.

E.g. you may want an entire (class) topos to be site for another.

The theorem shows (on some conditions) a sub*set* of that topos works as small site.

Published proofs by Verdier, or Mac Lane and Moerdijk, apply in MTT.

Not bald existence claims (to use Angus's phrase) but specified existence.

Theorem (Giraud theorem)

If a locally small category $\mathcal{E}$ has

a) *a limit for every finite diagram.*

b) *a coproduct for each set of sheaves, and these are stable disjoint unions.*

c) *a stable quotient for every equivalence relation.*

d) *a set $\{G_i | i \in I\}$ of generators.*

Then it is a Grothendieck topos.

A small site $\widetilde{\mathcal{C}}_J$ can be specified from the set of generators.

The derived category $D(X)$ starts with the category $\mathcal{K}(X)$ whose objects are complexes of quasi-coherent sheaves of modules over a scheme X

$$\cdots \longrightarrow M_{i-1} \longrightarrow M_i \longrightarrow M_{i+1} \longrightarrow M_{i+2} \longrightarrow \cdots$$

and arrows are *homotopy classes* of maps between complexes.

Complexes and homotopy classes are sets, provably existing in MC.

The derived category $D(X)$ is a certain calculus of fractions on $\mathcal{K}(X)$.

The key point conceptually and for MTT is that the definition of $D(X)_1$ depends on (infinitely many) complexes of small modules making (infinitely many) finite diagrams commute.

The complexes of modules are sets.

MTT proves there is a derived category $D(X)$, with a class of objects and collection of arrows.

By no means goes without saying that “set theoretic comprehension” in this sense suffices for all proofs in EGA and SGA.

But extensive checking, especially SGA 4 and 5, shows it is true.

Most of EGA and SGA is explicitly about “very small sets.”

This – not the comparison theorem with its specific conditions – justifies the common saying that we can always eliminate toposes in favor of small sites.

But more:

The set theoretically large structures of cohomology need only set theoretic comprehension.