"Nobody who understands these proofs does anything but think about very small structures from the start till the end."

– Expert quoted on F.O.M.

"Hendrik Lenstra twenty years ago was firm in his conviction that he did want to solve Diophantine equations, and did not not wish to represent functors—and now he is amused to discover himself representing functors in order to solve Diophantine equations!"

– Barry Mazur at 1995 BU conf. on FLT

"Very small" would mean at most continuum sized.

"Large" = the size of a naïve category of all groups, all sets, etc.

In ZFC that means the size of some *Grothendieck universe* modelling ZFC.

ZFC cannot prove Grothendieck universes exist.

"Grothendieck did not believe in universes. . . . He believed in toposes and schemes."

– Cartier and Bénabou discussion, ENS, 2005

For Grothendieck, schemes and toposes are "in close symbiosis."

The special and general cases of "espace nouveau style."

For him, both are large categories.

To “approach these categories from a ‘naïve’ point of view, [and] to avoid certain logical difficulties, we accept the notion of a Universe, a set ‘large enough’ that the habitual operations of set theory do not go outside it.”

– Grothendieck and Dieudonné and Grothendieck

Three specific reasons to use large sructures:

1. Ease of definition.
2. Relevance.
3. Concision.

Not generality per se, nor logcal strength.

Grothendieck describes a scheme as a “magic fan” combining many kinds of solution to each single polynomial problem.

The definition in Paris ca. 1954: whatever would work to prove the Weil conjectures.

Schemes as small sets:

Cf. Kronecker replacing $\sqrt{2}$ by polynomials in X, mod $X^2 - 2$.

X becomes a square root of 2: squaring and subtracting 2 gives 0.

Just so:

Given integer polynomial $P(X, Y, Z)$ in 3 variables, define a *scheme* $\mathrm{Spec}(\mathbb{Z}[X, Y, Z]/(P))$ as a "space of solutions" with $\langle X, Y, Z\rangle$ as generic solution.

Points of the scheme are *prime ideals* in $\mathbb{Z}[X, Y, Z]/(P)$.

They are entirely formal "solutions" to

$$P(X, Y, Z) = 0$$

just as Kronecker makes X "a solution" to $X^2 - 2$.

If this seems hard to get geometrically, Deligne is about to agree.

The polynomial rings in number theory are very small.

So these schemes are very small sets.

Schemes as functors of points.

Reconsider polynomial $P(X, Y, Z)$.

There are integer solutions to $P(X, Y, Z) = 0$, rational solutions, real solutions,... many kinds of solutions.

All are important – even if we only seek integer solutions!

Define the *functor of points* of $P(X, Y, Z)$ as a functor taking each ring R to the set of solutions of $P(X, Y, Z)$ in R

$$R \mapsto \{\langle a, b, c\rangle \in R^3 |\ P(a, b, c) = 0\}$$

As the category of all rings is large, this functor is large.

Generally do not use actually *all* rings.

But all algebras over some base ring.

Still large.

Why not just the specific rings we want to use?

$A.$ Specifyng them would be a pointlessly complicated task.

$B.$ Adds nothing to the basic theorems.

$C.$ It will change tomorrow.

The small set and large functor schemes of $P(X, Y, Z)$ contain the same information.

The *category of all small set schemes* is equivalent to the *category of all large functor schemes*.

Both those categories are (at least) large.

And the categories are what you want to know.

Deligne:

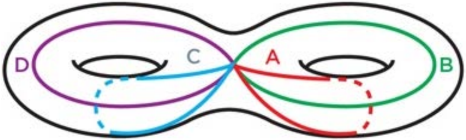
> *the points ... have no ready to hand geometric sense.... When one needs to construct a scheme one generally does not begin by constructing the set of points.... [While] the decision to let every commutative ring define a scheme gives standing to bizarre schemes, allowing it gives a category of schemes with nice properties.*

One generally constructs a scheme by (categorical) relations to other schemes.

Grothendieck representability theorem. Hilbert schemes.

One key “nice property” is nice definitions of *cohomology*.

Cohomology originally counted the holes in a Riemann surface by comparing integrals of one form along different routes.



In general, integrating on A, B, C, D all give different results.

Any closed path gives some combination of those results.

This surface has four (1-dimensional) holes, with many analytic consequences via deep theorems like Riemann-Roch.

Scheme cohomology

A *sheaf* $\mathscr{F}$ on a scheme X poses a problem all over X.

It may be easy to solve the problem in one small region of the scheme – say the region *modulo some power of 5*.

As in topology and analysis, cohomology relates such *local solutions* to *global* (i.e. actual) solutions.

I will not explain that in detail!

Surprising facts:

Cohomological intuition developed in topology gives insight here.

The *functorial* approaches to earlier cohomology worked here.

Grothendieck characterizes *sheaves* and their *cohomology* not by their internal nuts and bolts.

But by their (large) functorial relations.

The nuts and bolts come in later, only when and as they are needed for particular calculations.

Deligne makes the same point about motives: Grothendieck does not seek to define motives piecemeal by their nuts and bolts, but by their intrinsic relations in a category of motives.

Grothendieck later describes his *Tohoku* 1957 as handling

the "prodigious arsenal" of sheaves on X by "its most obvious structure, which appears so to speak right in front of your face, which is to say the structure of a category."

A large category.

Cohomology is a (large) universal δ-functor.

Neatly captures an idea familiar to Rieman long ago: locally compatible local solutions may not be globally compatible.

All variants of Grothendieck duality (for coherent cohomology) being developed today follow Grothendieck's approach:

A certain functor Rf_* between *derived categories* has a right adjoint $Rf^!$, with some further properties under some conditions on f.

These are large, or larger than large, depending on details.

Not all *applications* of Grothendieck duality use all of this.

Deligne: “Miraculously, the same formalism applies in étale cohomology, with quite different proofs.”

Deligne uses it for étale Poincaré duality in SGA 4 and 4.5.

A propos of SGA 4.5.

The book explicitly uses set-theoretically large sites. (Nb. *gros* and *petit* sites are both set theoretically large.)

The goal is to be "clearer than SGA 4 ... but not claim to give a complete proof" of the Weil conjectures.

Deligne relies on his own proofs of Poincaré duality, the trace formula, and more, in SGA 4, which explicitly use universes

SGA 4.5 offers a working background for Deligne's proof of the last Weil conjecture based cohomology of topological spaces plus “un peu de foi" (p. 1).

I.e. faith in the large structure proofs he cites without repeating them here.