My sister had this vacation (1925) in mind when she said a mountain landscape impressed the notion of purity upon her soul. Of course no such notion entered my head.... Distant shafts of sunlight crisscrossing at sundown gave me the idea of composing on several planes.... so the reader's mind is drawn behind the immediate object, toward more distant perspectives. – André Weil

I. André Weil

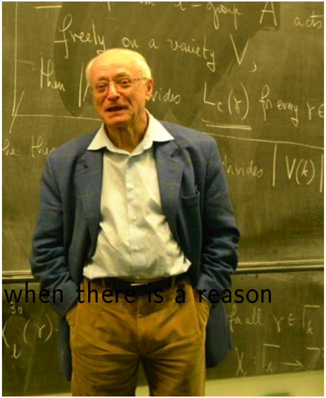
- A. Utterly unified view of mathematics.
- B. Disdains purity.
  - 1. Cohomological insight without cohomology.
- C. Uninterested in issues of logical strength.

II. Jean-Pierre Serre

A. “The incarnation of elegance” (AG).

B. “Serre made the cohomological approach to algebro-geometric questions explicit” (Mumford).

C. Occasional interest in logic. Interested when there is a reason for him to be.



III. Alexander Grothendieck

A. "The Grothendieck revolution," unthinkable without Serre.

B. Grothendieck calls it a "rebirth of geometry."

C. Aggressively interested in logical rigor.

D. Profligate of logical strength.

A great many people made it all work.

The most glaring omission in this talk is Deligne.

listening to Serre

In Serre's words Weil 1928 dealt with:

Diophantine equations, that is to say, with rational points on algebraic varieties.

At that time, the only known method was Fermat's descent; very often, the application of this method depended on explicit calculations, so that a different little miracle seemed to happen in each particular case.

Weil was the first to see that behind these computations there was a general principle . . . a sort of transfer between algebra (in principle easy) and arithmetic (harder).

Weil adopts the geometric idea from Hilbert and Hurwitz (1890) and independently Poincaré (1901).

In such hands, geometry "gives simple means to completely resolve . . . a crowd (*foule*) of particular problems" (Weil 1928).

Weil remarks that this approach identifies new crucial features: *genus* of a curve and more.

Weil himself unifies the little arithmetic miracles.

As Weil walked around Lanslevillard:

“I would often stop to open a notebook.

My calculations showed me that Fermat’s methods, as well as his successors’, all rested on one virtually obvious remark, to wit:

If $P(x, y)$ and $Q(x, y)$ are homogeneous polynomials algebraically prime to each other, with integer coefficients, and x, y are integers prime to each other, then $P(x, y)$ and $Q(x, y)$ are “almost" prime to each other . . .

their GCD admits a finite number of possible values.”

Polynomials $x + y$ and $x - 2y$ are algebraically prime.

For all relatively prime m, n: $\mathrm{GCD}(m + n, m - 2n)$ divides 3.

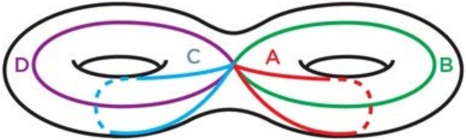
The simple proof for rational integer polynomials uses their splitting fields, which need not have unique prime factorization.

So it works over any number field in place of the rationals.

The generality comes unsought.

Weil: “I attempted to translate this remark into a birationally invariant language, and had no trouble doing so. Here already was the embryo of my future thesis.”

Weil 1928 does arithmetic via integration on Riemann surfaces.

$\Sigma_2 =$ 

Complex integration on the genus 2 Riemann surface Σ_2 is controlled by integration along the cycles A,B,C,D.

It becomes addition on the *Jacobian* $J(\Sigma_2)$, a 2 complex-dim torus, a quotient $\mathbb{C}^2/\Lambda$.

A genus n Riemann surface has Jacobian (analytically) an n complex-dim torus.

Weil confirms a conjecture by Poincaré that addition of points of any Jacobian defined over a fixed algebraic number field gives a finitely generated Abelian group.

Serre: "The algebraic geometry of the time had not yet developed the tools that were needed. Fortunately, Weil had read the works of Riemann at the École Normale, and he was able to replace the missing algebra by analysis: theta functions."

Weil was utterly uninterested in separating algebra/arithmetic from analysis.

Bourbaki co-founder Chevalley was passionately interested already in the 1920s (see *New Dictionary of Sci. Bio.* on Chevalley).

Weil calls Chevalley (1951 Algebraic Functions of One Variable) “algebra with a vengeance; algebraic austerity could go no further.... valuable and useful ... [yet] severely dehumanized."

Both Weil and Chevalley understood the project in terms of purity of method, not logical strength.

Weil 1928 says of theta functions:

"Sur les propriétés, utilisées ici, des fonctions thêta, v. p. ex. Krazer-Wirtinger, Enzykl. d. math. Wiss. II, B. 7."

Waves his hand at a 271 page encyclopedia article.

He neither knows nor cares to know which results he used.

To be clear: he knows all the results in the article. Does not care which are implicit in his thesis.

Weil went much farther connecting geometric analysis to arithmetic.

Serre describes masterful cases.

In some, "Weil had to be content with partial results, partly unproved but which would turn out to be essentially correct; a fortiori, he could make no arithmetic application. . . .his work on Riemann-Roch served as a model to others fifteen years later."

Where analysis did work, Weil would not *purify* it away.

The Weil conjectures.

$$Z(t) \;=\; \exp\,(\sum_{s=1}^{\infty} N_s \frac{t^s}{s}) = \;\frac{P_1(t)P_3(t)\cdots P_{2n-1}(t)}{P_0(t)P_2(t)\cdots P_{2n}(t)}$$

Bravura union of Eulerian analysis, Hilbert-Zariski bi-rational geometry, Betti-Lefschetz topology.

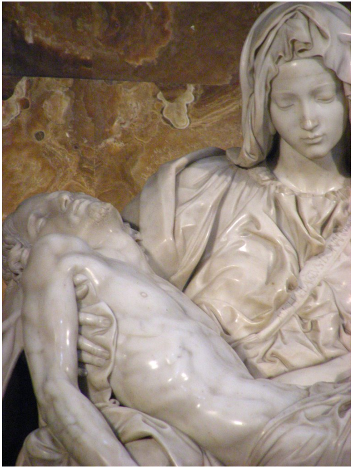
Cohomological conception/strategy using fixed points to count points on finite spaces.

His conjectures have a “really elementary character” which “does not appear clearly,” even in the analytic work by Hasse and Davenport that he uses for them.

All evidence, including statements by Serre and Grothendieck, says Weil did not believe these spaces could actually have cohomology.

Weil does not even keep “pure” of suggestive ideas he thinks will not actually work.

Serre believed in this cohomology, and convinced Grothendieck.



Imagine standing in front of Michaelangelo's Pietà, with Michaelangelo, and asking "What were you aiming for?"

Not a good question.

(made by the method of hammer and chisel)

I made such a mistake by asking Serre:

“What was your general idea when you found isotrivial maps?”

He said he has no general ideas.

One specific idea was that isotrivial covers give the 1-dim Weil cohomology.

$$\{1\} \to \Gamma(X, H) \longrightarrow \Gamma(X, G) \longrightarrow \Gamma(X, G/H) \longrightarrow \tilde{\mathrm{H}}^1(X, \underline{H}) \longrightarrow \tilde{\mathrm{H}}^1(X, \underline{G}) \longrightarrow \tilde{\mathrm{H}}^1(X, \underline{G/H})$$

Espaces fibrées algébriques, Seminaire Chevalley 1958

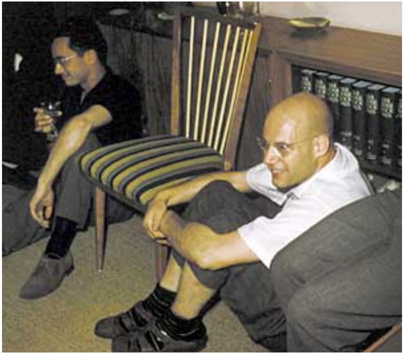
Swinnerton-Dyer: “one now instinctively assumes all obstructions are best described in terms of cohomology groups.’

Many earlier sources: Poincaré, Lefschetz, Noether, Alexandroff-Hopf, Weil, Eilenberg, Mac Lane, Cartan.

For Weil, cohomology captures dimension, transversality, and intersection number.

For Serre, one expects dualities in cohomology. Exact sequences.

For, Grothendieck, “a rebirth of geometry.”



AG and Serre 1958

When vast territories are being opened up, nothing could he more harmful to the progress of mathematics than a literal observance of strict standards of rigour. . . . At the same time, it should always be remembered that it is the duty, as it is the business, of the mathematician to prove theorems, and that this duty can never be disregarded for long without fatal effects.

Andre Weil Foundations of Algebraic Geometry 1946