

Are Directed Waves Multifractal?

Yi-Kuo Yu^{1,2} and H. Mathur¹

¹*Department of Physics, Case Western Reserve University, Cleveland, Ohio 44106-7079*

²*Department of Physics, Florida Atlantic University, 777 Glades Road, Boca Raton, Florida 33431*

(Received 11 February 1997; revised manuscript received 2 June 1998)

Wave propagation is studied in a sufficiently anisotropic random medium where backscattering along one direction can be neglected. A Fokker-Planck equation, the solution to which would provide a complete statistical description of such directed waves, is derived. The Fokker-Planck equation is mapped onto an $su(1,1)$ ferromagnet and its symmetries are identified. Using the symmetries, asymptotic wave function distributions are computed and used to show that directed wave functions fill space uniformly and do *not* have multifractal character. [S0031-9007(98)07511-5]

PACS numbers: 71.55.Jv, 05.40.+j, 72.15.Rn, 75.10.Jm

Much progress has been achieved in the study of wave propagation in random media by the infusion of scaling ideas from statistical mechanics and critical phenomena. Dirty electronic systems are the best studied example [1,2]. They are now known to exhibit a variety of phases and transitions, many of which remain poorly understood. Recently there has been interest in the application of multifractal analysis to these problems [2]. Fractals are complex geometric forms that exhibit self-similarity under magnification [3]. Multifractals are still more complex objects which need an infinite set of exponents to characterize their self-similar scaling rather than the single exponent that suffices for an ordinary fractal [4]. These ideas will be expressed precisely below; no previous background in multifractal analysis is needed to read this paper.

In statistical mechanics the typical configuration of a disordered spin system at a critical point is complicated and for certain models can be shown to exhibit multifractal scaling [5]. The purpose of such an analysis is to provide a characterization of the critical point in terms of multifractal exponents that supplement the conventional universal exponents of critical phenomena. Dirty electronic wave functions have also been studied from this perspective. For example, extensive numerical simulations indicate that the wave functions of a quantum Hall system are multifractal at the critical point that separates two Hall resistance plateaus [2]. Calculation of multifractal exponents requires detailed statistical information about the wave functions and analytic results are therefore difficult to obtain. With few exceptions [6], multifractal analysis for dirty electronic systems is therefore carried out via numerical simulation. In this Letter we report multifractal analysis of directed wave propagation through a random medium. This problem has been investigated by the authors of Ref. [7] who claimed numerical evidence of multifractal scaling.

The directed wave model is very rich in physical applications. It applies whenever waves propagate in a sufficiently anisotropic medium where backscattering in one direction can be neglected [8,9]. An experimental realization [10,11] that has attracted much attention recently

([12–15], and references therein) is the surface of a quantum Hall multilayer. Studies of directed waves have a long history due to the equivalence of this problem to time-dependent wave propagation in a noisy environment [7,16]. Moreover, a recent model for stress propagation in granular materials [17] is identical to lattice models of directed waves [18,19]. Still another motivation for studying the directed wave model has been its formal similarity to the directed polymer problem [20] (for an illuminating discussion of this viewpoint see, for example, Ref. [18]).

Previous analytic work on directed waves has focused on the calculation of low-order correlations of the wave functions (needed, for example, to calculate observables such as the disorder averaged conductance of a quantum Hall multilayer). Here it is necessary to analyze the full probability distribution of the wave functions. Our method is to derive a Fokker-Planck equation that governs the evolution of the probability distribution. This equation has an $su(1,1)$ symmetry that can be exploited to obtain the asymptotic probability distribution which reveals that directed waves fill space quite uniformly and therefore do not change appearance under magnification. Thus, contrary to previous numerical work, we are able to show exactly that directed wave functions do *not* have a multifractal character. The asymptotic wave function distribution [Eq. (3) below] is the directed wave analog of a result derived several years ago for directed polymers in one dimension by Huse *et al.* [21].

We shall take directed waves at a fixed frequency to evolve according to (see, for example, Refs. [13,14])

$$-i \frac{\partial}{\partial x} \psi_n(x) + t_n(x) \psi_{n+1}(x) + t_{n-1}^*(x) \psi_{n-1}(x) = 0. \quad (1)$$

Backscattering is neglected along the x direction. The transverse direction is taken to be discrete and the system is assumed to be of finite size, N , in this direction (with periodic boundary conditions). The anisotropic and directional nature of Eq. (1) is reflected in the fact that it is of first order in the x direction, whereas it is of second order in the transverse direction. Disorder is incorporated

by taking the hopping terms $t_n(x)$ to be random with statistics given below. Other models have been considered in the literature. They are either equivalent to Eq. (1) or are believed to have the same qualitative behavior [22]. The problem posed is the following: The wave function is specified for a fixed value of x ; it is then evolved in the x direction according to Eq. (1). For definiteness, it may be supposed that the wave function is localized at $n = 0$ at $x = 0$. Thus $\psi_n(x = 0) = \delta_{n,0}$. Each disorder realization will produce a different wave function at a larger value of $x > 0$. We are interested in $P(a_n, x)$, the proba-

bility density that at a fixed $x > 0$ the wave function is $\psi_n(x) = a_n$.

Our problem resembles the problem of Brownian motion, in which a heavily damped particle is subject to a noisy environment. Standard methods exist in the theory of Brownian motion to convert the stochastic equation of motion (called the Langevin equation) into the corresponding Fokker-Planck equation which governs the time evolution of the probability density of the particle position [23]. These methods may be applied to obtain the Fokker-Planck equation corresponding to Eq. (1),

$$-\partial P(a_n, x)/\partial x = \mathcal{H}P,$$

$$\begin{aligned} \mathcal{H} = & D \sum_{\alpha, m} a_m^\alpha \frac{\partial}{\partial a_m^\alpha} - \frac{D}{4} \sum_{\alpha, \beta, m} \left[(a_{m+1}^\alpha)^2 \frac{\partial^2}{\partial a_m^{\beta 2}} + (a_m^\alpha)^2 \frac{\partial^2}{\partial a_{m+1}^{\beta 2}} \right] + \frac{D}{2} \sum_{\alpha, \beta, m} a_m^\alpha \frac{\partial}{\partial a_m^\alpha} a_{m+1}^\beta \frac{\partial}{\partial a_{m+1}^\beta} \\ & - \frac{D}{2} \sum_{\alpha, \beta, \rho, \nu, m} \epsilon_{\alpha, \beta} \epsilon_{\rho, \nu} a_m^\alpha \frac{\partial}{\partial a_m^\beta} a_{m+1}^\rho \frac{\partial}{\partial a_{m+1}^\nu}, \end{aligned} \quad (2)$$

where the Greek letters represent either 1 or 2. Here we have refined our notation to write $a^1 =$ real part of the wave function and $a^2 =$ imaginary part. Likewise, t^1 and t^2 denote the real and imaginary parts of the hopping term. In deriving Eq. (2) it was assumed that the hopping is a Gaussian white noise process with zero mean and variance given by $[t_n^\alpha(x) t_m^\beta(x')]_{\text{ens}} = (D/2) \delta_{\alpha, \beta} \delta_{n, m} \delta(x - x')$. $[\dots]_{\text{ens}}$ denotes an average over the ensemble of disorder realizations. Given an initial wave function, $\psi_n(x = 0)$, the Fokker-Planck Eq. (2), in principle, allows the calculation of the complete probability distribution of the wave function, $P(a, x)$ for larger values of x . In practice, this may appear difficult because Eq. (2) is a partial differential equation in a large number ($2N + 1$) of variables. However, it will be seen that Eq. (2) has a number of helpful symmetries.

First, note that the stationary solutions to the Fokker-Planck equation must be radial. It is easy to verify by substitution that if P is radial [a function only of $\sum_{n, \alpha} (a_n^\alpha)^2$ and possibly x] then $\partial P/\partial x = 0$, showing that radial solutions are stationary. The converse, that stationary solutions *must* be radial, is also true and will be shown below. It is a consequence of an $\text{su}(1,1)$ symmetry of the Fokker-Planck equation to which we shall return.

A second useful property of Eq. (2) follows from probability conservation. By direct substitution into the wave Eq. (1), it can be shown that the total probability $\sum_{n, \alpha} (\psi_n^\alpha)^2$ does not change with x for any disorder realization. Consequently, if we solve the Fokker-Planck equation subject to the initial condition that the system begins with some definite normalized wave function, P must live on the unit sphere in a space. In other words, P vanishes unless $\sum_{n, \alpha} (a_n^\alpha)^2 = 1$.

The only stationary distribution consistent with probability conservation is a uniform distribution on the unit sphere in a space. Thus the large x asymptotic distribution

is uniquely determined to be

$$P(a, \infty) = \frac{1}{A} \delta \left(\sum_{n, \alpha} (a_n^\alpha)^2 - 1 \right). \quad (3)$$

Here A is a constant fixed by the normalization condition $\int \prod_{n, \alpha} da_n^\alpha P(a, \infty) = 1$. The physical content of Eq. (3) is that directed wave functions are completely randomized after propagating a large distance; however, they remain normalized. Precisely the same distribution arises in connection with the distribution of matrix elements of correlated random matrices [24]; thus techniques for averaging over such distributions are well developed.

Multifractal characterization of the directed wave functions begins with the calculation of the exponents $\zeta(q)$ which are defined via [7]

$$\left[\sum_n \left(\sum_\alpha |\psi_n^\alpha|^2 \right)^q \right]_{\text{ens}} \sim N^{\zeta(q)}. \quad (4)$$

The generalized fractal dimensions $D(q)$ are related to these exponents by $\zeta(q) \equiv (1 - q)D(q)$. For $q = 2$, the left-hand side of Eq. (4) is the inverse of the participation ratio which has long been studied in connection with disordered electronic systems [25]. Roughly, it measures the number of sites on which the wave function has a substantial weight. Equation (4) shows that $\zeta(q)$ characterizes the growth of generalized inverse participation ratios with system size. Evidently, $\zeta(0) = 1$ and $\zeta(1) = 0$ (by normalization). To calculate the multifractal dimensions for other values of q , it is necessary to average the inverse participation ratios over the wave function distribution in Eq. (3). Performing this average for large N yields

$$\left[\sum_n \left(\sum_\alpha |\psi_n^\alpha|^2 \right)^q \right]_{\text{ens}} \simeq \Gamma(q + 1) N^{1-q}. \quad (5)$$

Comparing Eqs. (4) and (5), we see that $\zeta(q) = 1 - q$ [which implies $D(q) = 1$]. This is precisely the scaling expected of an object that is *not* multifractal.

To make contact with previous numerical work [7] it is necessary to calculate the $g(\beta)$ spectrum which is defined as follows [4,7]: Let w be the weight (modulus square of the wave function) on a particular site. β is a logarithmic measure of the weight defined as $\beta \equiv \ln w / \ln N$. Assemble a histogram of β values by drawing from the ensemble of disorder realizations and let $\Pi_N(\beta')d\beta$ represent the frequency with which β lies between β' and $\beta' + d\beta$ in a system with N sites. For large N it is expected that $\Pi_N(\beta) \sim N^{g(\beta)}$, which defines $g(\beta)$. To calculate $g(\beta)$ it is helpful to first evaluate $P(w)dw$, the probability that the weight on a particular site lies between w and $w + dw$, obtained by integrating the distribution in Eq. (3) over all sites except one. For large N the result is $P(w) = N \exp(-Nw)$. Straightforward substitution into the definition above then yields

$$g(\beta) = 2 + \beta \quad (6)$$

for $\beta < -1$. Note $g(\beta) > 0$ over the range $-2 < \beta < -1$.

In Ref. [7] the $g(\beta)$ spectrum is calculated numerically and found to have support on the interval $-1.97 < \beta < -0.88$. The authors of Ref. [7] assert (incorrectly) that, if $g(\beta)$ has support over a range away from $\beta = -1$, it necessarily implies multifractal scaling. This is the primary

$$\mathcal{H} = \frac{D}{2} \sum_n \left[2 \sum_{\alpha} b_n^{\alpha\dagger} b_n^{\alpha} + \sum_{\alpha,\beta} b_n^{\alpha\dagger} b_n^{\alpha} b_{n+1}^{\beta\dagger} b_{n+1}^{\beta} - \frac{1}{2} \sum_{\alpha,\beta} (b_n^{\alpha\dagger} b_n^{\alpha\dagger} b_{n+1}^{\beta} b_{n+1}^{\beta} + b_n^{\alpha} b_n^{\alpha} b_{n+1}^{\beta\dagger} b_{n+1}^{\beta\dagger}) + \sum_{\alpha,\beta,\rho,\nu} \epsilon_{\alpha,\beta} \epsilon_{\rho,\nu} b_n^{\alpha\dagger} b_n^{\beta} b_{n+1}^{\rho\dagger} b_{n+1}^{\nu} \right]. \quad (8)$$

Thus the Fokker-Planck equation is seen to be equivalent to an interacting boson problem.

The symmetry of the Hamiltonian is revealed by considering the algebra of the bosonic bilinear terms out of which it is built. Define $K_n^+ \equiv \frac{1}{2} \sum_{\alpha} b_n^{\alpha\dagger} b_n^{\alpha\dagger}$, $K_n^- \equiv \frac{1}{2} \sum_{\alpha} b_n^{\alpha} b_n^{\alpha}$, $K_n^z \equiv \frac{1}{2} (b_n^{1\dagger} b_n^1 + b_n^2 b_n^{2\dagger})$, and $M_n \equiv i(b_n^{1\dagger} b_n^2 - b_n^{2\dagger} b_n^1)$. These operators obey

$$[K^+, K^-] = -2K^z, \quad [K^z, K^{\pm}] = \pm K^{\pm}, \quad (9)$$

and M commutes with all of the K 's. In writing Eq. (9), the site indices have been suppressed for clarity. These commutation relations apply only if the indices coincide. Operators with distinct indices all commute. This algebra is reminiscent of the angular momentum algebra—the notation was chosen to highlight the similarity. However, there is a crucial sign difference in the $[K^+, K^-]$ commutator. Thus the \vec{K} operators actually obey the $\text{su}(1,1)$ algebra rather than $\text{su}(2)$. $\text{su}(1,1)$ is a well-studied classical Lie algebra sometimes called the *hyperbolic* angular momentum algebra in the physics literature when it is discussed in connection with Schwinger's coupled-boson description of angular momentum [26].

The Hamiltonian may be rewritten in terms of these operators as an $\text{su}(1,1)$ ferromagnet [a generalization of

basis for their claim of multifractal scaling since they infer that the actual deviation from ordinary scaling is very small: $\zeta(q)$ (numerical) $= 1 - q + 3.6 \times 10^{-3} q^2$. The asymptotic distribution derived here [Eq. (3)] explicitly shows both ordinary scaling and a spectrum, $g(\beta)$, with an extended support away from $\beta = -1$. Moreover, the form of $g(\beta)$ we obtain [Eq. (6)] provides a good fit to the numerical data (Fig. 3 of Ref. [7]). Hence we believe that the numerical data of Ref. [7] are in fact consistent with our conclusion that directed waves are not multifractal.

To complete the analysis we must now return to the Fokker-Planck equation and show that all stationary solutions are radial. Note that Eq. (2) has the appearance of an imaginary time Schrödinger equation. From this point of view $P(a, x)$ is the wave function, and we are interested in the multiplet of eigenfunctions of the “Hamiltonian,” \mathcal{H} , with eigenvalue zero. It is useful to express \mathcal{H} in a second quantized language by introducing harmonic oscillator ladder operators

$$b_n^{\alpha} = \frac{1}{\sqrt{2}} \left(a_n^{\alpha} + \frac{\partial}{\partial a_n^{\alpha}} \right), \quad (7)$$

which obey the bosonic commutation relations $[b_n^{\alpha}, b_m^{\beta\dagger}] = \delta_{\alpha\beta} \delta_{nm}$ and $[b_n^{\alpha}, b_m^{\beta}] = 0$. In this language the Hamiltonian is given by

the ordinary Heisenberg ferromagnet in which $\text{su}(1,1)$ operators replace their spin counterparts] [27]:

$$\mathcal{H} = D \sum_n \left[2K_n^z K_{n+1}^z - (K_n^+ K_{n+1}^- + K_n^- K_{n+1}^+) - \frac{1}{2} (M_n M_{n+1} + 1) \right]. \quad (10)$$

It is now easy to verify that the Hamiltonian is invariant under $\text{su}(1,1)$ rotations:

$$[\mathcal{H}, \vec{K}_{\text{tot}}] = 0, \quad (11)$$

where $\vec{K}_{\text{tot}} \equiv \sum_n \vec{K}_n$.

The complete set of eigenstates with eigenvalue zero may now be found. Because of the symmetry expressed in Eq. (11), this multiplet should form an irreducible representation of the $\text{su}(1,1)$ algebra. Note that the boson vacuum is a zero energy eigenstate of the Hamiltonian. This is evident from inspection of Eq. (8). In addition, the vacuum is annihilated by K_{tot}^- . Thus the vacuum is the lowest weight state in the multiplet; the complete infinite set may be obtained by repeatedly applying K_{tot}^+ to the vacuum [26].

To see that each member of the multiplet $(K_{\text{tot}}^+)^l|0\rangle$ with $l = 0, 1, 2, \dots$ corresponds to a radial function it is necessary to translate back into the first quantized language. The ground state of a harmonic oscillator has a Gaussian wave function; hence the boson vacuum is given by

$$|0\rangle \rightarrow \exp\left(-\frac{1}{2} \sum_{n,\alpha} (a_n^\alpha)^2\right), \quad (12)$$

a radial function. K_{tot}^+ is given by

$$K_{\text{tot}}^+ \rightarrow \frac{1}{4} \sum_{n\alpha} \left[(a_n^\alpha)^2 + \frac{\partial^2}{\partial a_n^{\alpha 2}} - 2a_n^\alpha \frac{\partial}{\partial a_n^\alpha} - 1 \right]. \quad (13)$$

It is easy to verify that the action of K_{tot}^+ on *any* radial function will yield another radial function. Thus we have shown that the vacuum is radial as is any state built out of it by repeated application of the raising operator K_{tot}^+ ; in other words, a stationary solution must be radial.

Finally, note that, although in this paper we have focused on stationary solutions, it may be possible to give a rather complete analysis of the directed wave problem due to the equivalence demonstrated here to a relatively simple model—an $\text{su}(1,1)$ ferromagnet.

In summary, the main results of this paper are derivation of a Fokker-Planck equation that governs the wave function distribution for directed waves, mapping of this equation onto an $\text{su}(1,1)$ spin chain which reveals its symmetries, and calculation of the asymptotic wave function distribution which shows that directed waves fill space quite uniformly and are not multifractal.

We thank Leon Balents, Sue Coppersmith, Matthew Fisher, and Onuttom Narayan for discussions, and especially David Huse for pointing out that the $g(\beta)$ spectrum for our distribution is consistent with the data of Ref. [7]. H. Mathur thanks Matthew Fisher for hospitality at the ITP, Santa Barbara, where this work was initiated. H. Mathur is supported by NSF Grant No. DMR 98-04983 and by the Alfred P. Sloan Research foundation; the work at Santa Barbara was supported via NSF Grant No. PHY 94-07194. Yi-Kuo Yu was supported by the National Science Foundation at the ALCOM Science and Technology Center funded by Grant No. DMR 89-20147 and by startup funds from Florida Atlantic University.

- [1] P. A. Lee and T. V. Ramakrishnan, *Rev. Mod. Phys.* **57**, 287 (1985).
- [2] B. Huckestein, *Rev. Mod. Phys.* **67**, 357 (1995).
- [3] B. B. Mandelbrot, *The Fractal Geometry of Nature* (Freeman, San Francisco, 1983).
- [4] T. C. Halsey, M. H. Jensen, L. P. Kadanoff, I. Procaccia, and B. I. Shraiman, *Phys. Rev. A* **33**, 1141 (1986).
- [5] A. W. W. Ludwig, *Nucl. Phys.* **B330**, 639 (1990).
- [6] A. W. W. Ludwig *et al.*, *Phys. Rev. B* **50**, 7526 (1994); I. I. Kogan, C. Mudry, and A. M. Tsvelik, *Phys. Rev. Lett.* **77**, 707 (1996); C. Chamon, C. Mudry, and X. G. Wen, *Phys. Rev. Lett.* **77**, 4194 (1996); H. E. Castillo *et al.*, *Phys.*

- Rev. B* **56**, 10 668 (1997); L. Balents and M. P. A. Fisher, *Phys. Rev. B* **56**, 12 970 (1997).
- [7] J. P. Bouchaud, D. Touati, and D. Sornette, *Phys. Rev. Lett.* **68**, 1787 (1992).
- [8] L. Golubovic, S. Feng, and F. Zeng, *Phys. Rev. Lett.* **67**, 2115 (1991).
- [9] S. Feng, L. Golubovic, and Y.-C. Zhang, *Phys. Rev. Lett.* **65**, 1028 (1990).
- [10] H. L. Stormer *et al.*, *Phys. Rev. Lett.* **56**, 85 (1986).
- [11] D. P. Druist *et al.*, *Phys. Rev. Lett.* **80**, 365 (1998).
- [12] J. T. Chalker and A. Dohmen, *Phys. Rev. Lett.* **75**, 4496 (1995); L. Balents and M. P. A. Fisher, *Phys. Rev. Lett.* **76**, 2782 (1996); Y. B. Kim, *Phys. Rev. B* **53**, 16 420 (1996); Yi-Kuo Yu, cond-mat/9611137.
- [13] L. Balents, M. P. A. Fisher, and M. R. Zirnbauer, *Nucl. Phys.* **B483**, 601 (1996).
- [14] H. Mathur, *Phys. Rev. Lett.* **78**, 2429 (1997).
- [15] Ilya A. Gruzberg, N. Read, and Subir Sachdev, *Phys. Rev. B* **55**, 10 593 (1997); *Phys. Rev. B* **56**, 13 218 (1997).
- [16] A. A. Ovchinnikov and N. S. Erikhman, *Zh. Eksp. Teor. Fiz.* **67**, 1474 (1974) [*Sov. Phys. JETP* **40**, 733 (1974)]; A. Madhukar and W. Post, *Phys. Rev. Lett.* **39**, 1424 (1977); S. M. Girvin and G. D. Mahan, *Phys. Rev. B* **20**, 4896 (1979).
- [17] C. H. Liu *et al.*, *Science* **269**, 513 (1995).
- [18] L. Saul, M. Kardar, and N. Read, *Phys. Rev. A* **45**, 8859 (1992).
- [19] R. Friedberg and Y. K. Yu, *Phys. Rev. E* **49**, 5755 (1994).
- [20] A.-L. Barabasi and H. E. Stanley, *Fractal Concepts in Surface Growth* (Cambridge University Press, New York, 1995).
- [21] D. A. Huse, C. L. Henley, and D. S. Fisher, *Phys. Rev. Lett.* **55**, 2924 (1985).
- [22] For example, one could include a random on-site term $V_n(x)\psi_n(x)$ in Eq. (1). However, this term can be gauge transformed into a hopping term leading to Eq. (1), and in the strong disorder limit the statistics of the induced hopping term remain essentially white noise; see, for example, Ref. [13]. Another possibility is to discretize time as in Ref. [18].
- [23] *Selected Papers on Noise and Stochastic Processes*, edited by N. Wax (Dover, New York, 1954).
- [24] N. Ullah, *Nucl. Phys.* **58**, 65 (1964); M. L. Mehta, *Random Matrices* (Academic, New York, 1967), Chap. 15.
- [25] D. J. Thouless, *Phys. Rep. C* **13**, 93 (1974).
- [26] J. Schwinger, in *Quantum Theory of Angular Momentum*, edited by L. C. Biedenharn and H. Van Dam (Academic, New York, 1965); D. Mattis, *The Theory of Magnetism I* (Springer-Verlag, Berlin, 1988). Although the term $\text{su}(1,1)$ algebra is not used in these references, explicit representations of $\text{su}(1,1)$ are constructed using harmonic oscillator ladder operators. Readers familiar with these references should note that their bosons are related to ours via a canonical transformation.
- [27] Calculation of low-order wave function correlations has previously been mapped onto a supersymmetric spin chain in Refs. [13,15] [which has a $U(2-1,1)$ supersymmetry group of which $SU(2)$ and $SU(1,1)$ are subgroups] and onto a Heisenberg ferromagnet in Ref. [14]. Here, in contrast, calculation of the *full* wave function distribution is mapped onto an $\text{su}(1,1)$ spin chain.