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On Feynman's analysis of the geometry of Keplerian orbits

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A geometrical construction, introduced by Maxwell and Feynman to demonstrate that closed Keplerian orbits are elliptical, is adapted to show that open Keplerian orbits are hyperbolic or parabolic.
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I. INTRODUCTION

The publication of Newton's *Principia Mathematica* in 1687 was a major landmark in the history of science.[1] In Sec. III of his book Newton deduced Kepler's empirical laws of planetary motion from his law of gravitation and laws of motion. This derivation is very different from the analytical method now taught in mechanics courses. Newton's heavy use of geometry, including many unfamiliar propositions, makes his analysis nearly impenetrable to the modern reader. Recently, this approach motivated the noted gravitation theorist Chandrasekhar to write a volume of commentary.[2] In the 1960s, Newton's approach prompted Feynman to seek an alternative geometrical derivation of Kepler's law of ellipses that might be more accessible to contemporary readers. Feynman revealed part of this derivation in Ref. 3 and gave a full account of it in a lecture to Caltech undergraduates that was published posthumously.[4]

Interestingly, in the nineteenth century, Maxwell also appears to have sought a simplified geometric derivation of Kepler's law of ellipses. A proof, identical in every respect to Feynman's, appears in his 1877 treatise *Matter and Motion*.[5]

After he dispatched closed elliptic orbits in the *Principia*, Newton gave a separate analysis of open hyperbolic and parabolic orbits ''...because of the dignity of the Problem and its use in what follows.''[6] However in the published accounts of their work, neither Maxwell nor Feynman applied their method to open orbits. We make that extension in this paper. We find that hyperbolic orbits yield very straightforwardly to the geometrical approach, but parabolic orbits require some modification of the construction.

The principal value of this geometric exercise is that it provides the reader with some insight into the problem solving in which Newton engaged while he made the great conceptual breakthroughs that laid the foundations of classical mechanics. Readers who wish to tackle the problem for themselves should read only Sec. II in which we review the essential parts of Feynman's lecture and formulate the problem of open orbits; the solution is provided in Secs. III, IV, and the Appendix.

II. ELLIPTIC MOTION

It is useful to first recall Feynman's analysis of elliptical orbits.[4] We will review in detail only those aspects of it that must be modified for parabolic orbits.

By an elementary argument, Feynman was able to show that in velocity space (a graph in which the x and y components of the planet's velocity are plotted along the x and y axis, respectively) a planet must move along a circular arc. Note that the circle is not necessarily centered at the origin of the velocity space. The origin may lie inside the circle, outside it, or on its circumference.[7]

Figure 1 illustrates another aspect of planetary motion revealed by Feynman's argument. Suppose that as the planet moves from a to b along its orbit in real space, it moves from A to B in velocity space. Then the angle between a and b as seen from the sun is the same as the angle between A and B at the center of the circle.

From these facts we can deduce the shape of the planetary orbit. Following Feynman, let us first consider the case where the origin is enclosed by the circle in velocity space. Figure 2(a) shows this case in velocity space; Fig. 2(b) shows a plausible guess (much aided by the benefit of hindsight) of the corresponding real space orbit. The decisive test of whether the curve drawn in Fig. 2(b) has the correct shape is that the tangent at each point on the orbit must be parallel to the velocity at that point. Thus the tangent at a must be parallel to OA, the tangent at b must be parallel to OB, the tangent at c must be parallel to OC, and so on.

Before setting out to systematically construct a curve that passes this test, it is helpful to rotate the velocity diagram clockwise by 90°. It is customary to call such a rotated velocity diagram a hodograph. The term was used by Maxwell in 1877;[5] he attributed it to Hamilton. Figure 3(a) shows the planetary hodograph; Fig. 3(b), the putative real space orbit. For it to be correct, the tangent at a should be perpendicular to OA; similarly, the tangent at b should be perpendicular to OB, and so on.

We now describe the geometric construction used by Feynman. Consider a circle with center at Q. Let O be another point in the interior of the circle and A be a point on the circumference (see Fig. 4). Construct the perpendicular bisector of the line segment OA, and let a be the point where the bisector intersects the radius QA. We consider a to be the ''image'' of A under this construction. If we repeat the construction while moving the point A around the circle, the curve traced out by a is an ellipse with O and Q as its foci. Moreover, the perpendicular bisector Ba is the tangent to this ellipse at a. These assertions will be proved below; first let us make use of them.

We identify the circle in Fig. 4 as the planetary hodograph with O the origin of the velocity space and Q the center of the velocity circle. If we also locate the sun at Q, then the dotted ellipse satisfies our above criteria: by construction it

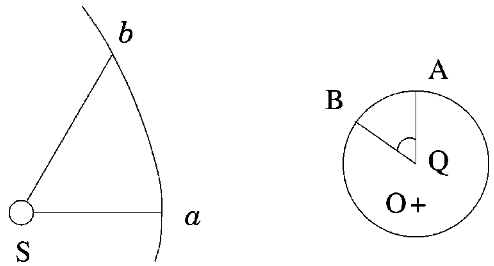

Fig. 1. As the planet moves from a to b along its orbit in real space, it moves from A to B in velocity space. Angle aSb equals AQB. Here S is the location of the sun in real space and Q is the location of the center of the circular orbit in velocity space.

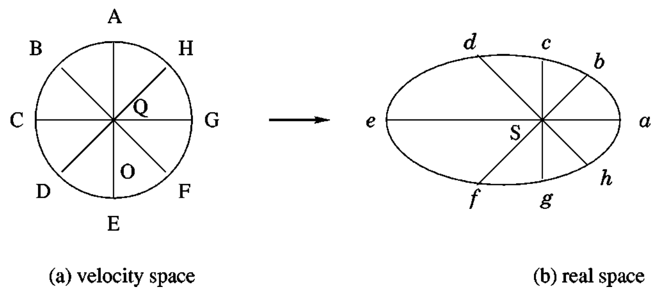

Fig. 2. (a) The orbit in velocity space. O is the origin in velocity space and Q is the center of the circle. Points are marked off at 45° intervals along the orbit. (b) The corresponding orbit in real space. As seen from the sun, S, the angular separation between the marked points must also be 45°.

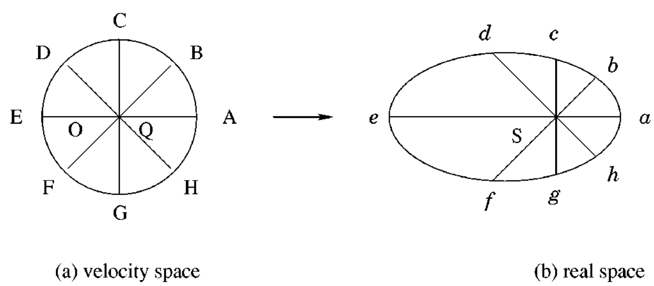

Fig. 3. Same as Fig. 2 except that the velocity space orbit has been rotated by 90°.

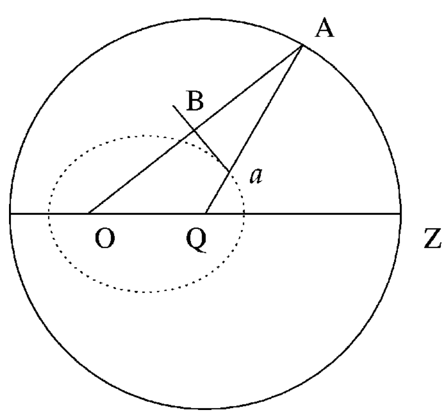

Fig. 4. *Feynman's diagram, Maxwell's construction*: A circle with center Q. O is a point in the interior of the circle, and A is an arbitrary point on the circumference. Ba is the perpendicular bisector of OA. As A moves around the circle, its image a traces an ellipse (dotted curve).

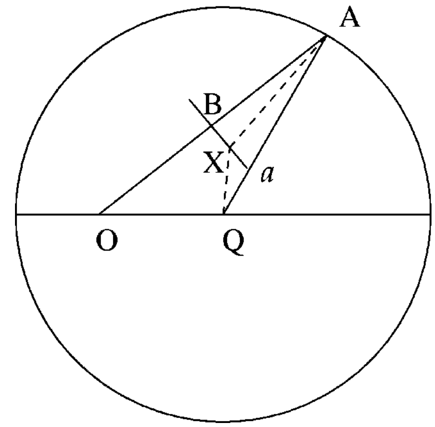

Fig. 5. The same as Fig. 4. X is an arbitrary point on the line Ba.

has the correct tangent at each point. Thus we have shown that planetary orbits that enclose the origin in velocity space are ellipses with the sun at one focus in real space. This statement is precisely Kepler's law of ellipses.

To complete the analysis we must prove our assertions regarding the dotted curve in Fig. 4. Note that the triangles OBa and ABa are congruent because they have a common side aB; moreover, by construction $OB=BA$ and $\angle OBa = \angle ABa = 90°$. It follows that $Oa+Qa=Qa+Aa$ equals the radius of the circle. Thus we have shown that for any point on the dotted curve, the sum of its distance to the points O and Q has a fixed value equal to the radius of the circle. Therefore, by definition the curve is an ellipse with O and Q its foci and its major axis equal to the radius of the circle.

Next we must demonstrate that Ba is the tangent at a. To this end we will show that for any point X on the extended line Ba, $OX+QX$ is greater than the major axis of the ellipse (see Fig. 5). All points X therefore lie outside the ellipse and the line Ba is therefore a tangent. To make this argument note that the triangles OBX and ABX are congruent and hence $OX=AX$. The triangle inequality applied to the triangle QAX states that $QX+AX$ is greater than the radius of the circle which equals the major axis of the ellipse. Hence $QX+OX$ is greater than the major axis of the ellipse which was to be shown.

The arguments above conclude our review of Feynman's lecture. It remains to study the cases where the origin lies outside the circle in velocity space and where it lies on the circumference, which leads, respectively, to hyperbolic and parabolic orbits. In his lecture Feynman made a few remarks about hyperbolic orbits, but he did not give a full discussion of either of these cases; instead he elects to "leave some of these things for you to play with." Following this injunction we find that hyperbolic orbits yield quite straightforwardly to Feynman's construction shown in Fig. 4; however, it fails for parabolic orbits and must be modified. The modified construction for parabolic orbits is the principal new content of this paper.

III. HYPERBOLIC MOTION

Throughout this paper we will refer to the body moving in the gravitational field of the sun as a planet. Strictly speaking, this usage is incorrect; the term planet applies to large

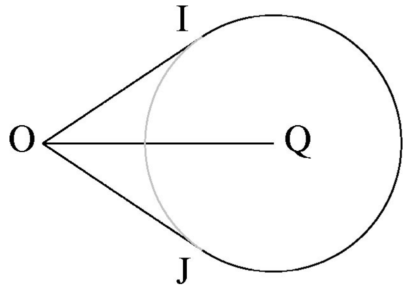

Fig. 6. A circle with center Q. Tangents from the exterior point O partition the circle into two arcs shown in gray and black.

objects that pursue elliptic orbits, not the open orbits analyzed in this section and the next. Open orbits are pursued by comets and a small number of deep space probes, among other objects. However, we shall continue with our imprecise usage because our focus is on gravitational physics, not solar system taxonomy.

Figure 6 shows a circle with center Q and O a point *exterior* to it. There are two tangents to the circle, IO and JO that pass through O. The points I and J divide the circle into two arcs shown in gray and black in Fig. 6.

Consider a point A on the black arc of the circle (see Fig. 7). Following Feynman's construction we draw the perpendicular bisector of OA. Let a be the "image point" at which the bisector intersects QA extended. As the point A is moved around the arc, the image point will trace out the dotted curve shown in Fig. 7. Note that the image points of I and J are at infinity. We shall prove below that this curve is a hyperbola with Q the near focus and O the far focus. Moreover Ba is a tangent to the hyperbola at a.

A second hyperbola is obtained by this construction if the point A is placed on the gray arc (Fig. 8). For this hyperbola O is the near focus and Q is the far focus. As before, Ba is a tangent. If we place the sun at Q, both hyperbolae are curves that have the correct tangent at each point. Which is the true shape of the orbit? We need additional physical input to resolve this question. If the orbit in Fig. 8 were the true orbit, we would have the strange situation that the planet moves slowest at the point of nearest approach to the sun and

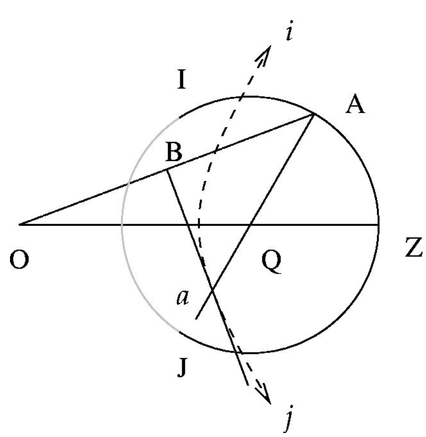

Fig. 7. The construction of Fig. 4 with O exterior to the circle and the point A confined to the black arc. Ba is the perpendicular bisector of OA. As A moves along its arc, its image under this construction traces a hyperbola.

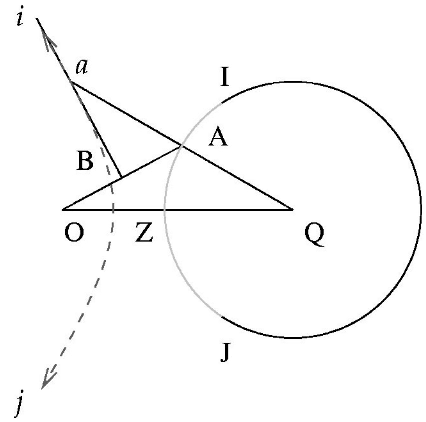

Fig. 8. Same as Fig. 7 except that A is confined to the gray arc.

speeds up as it moves away from the sun. Clearly this motion would violate energy conservation. Therefore the orbit in Fig. 7 must be the true orbit.

We do not have to invoke energy conservation to decide between the competing orbits. It is possible to adapt Newton's pictorial method of approximating the orbit as a series of straight segments to show that the orbit in Fig. 8 is unphysical. The essence of the argument is to show that if an object moves away from the sun it must slow down. It is sufficient to show this for an infinitesimal motion. However we shall not pause to make this argument here. In Feynman's words, "one should not ride in the buggy all the time." [4]

The orbit sketched in Fig. 8 is not completely unphysical however. It would describe the motion if the force were repulsive. It applies, for example, to the classical motion of an alpha particle scattering from a point nucleus—the problem analyzed by Rutherford and discussed from this geometrical point of view by Feynman.

To complete this section we must now prove the assertions made above. First note that in Fig. 7 the triangles OBa and ABa are congruent by construction just as their Fig. 4 counterparts were. It follows that $Oa-Qa=Aa-Qa$ equals the radius of the circle. Thus for any point on the dotted curve, its distance from O minus its distance from Q is a constant, the radius of the circle. Thus by definition, the dotted curve is a hyperbola with Q the near focus and O the far focus.

Now let us prove Ba is the tangent at a to the hyperbola. Again consider a point X on the extended line Ba (Fig. 9). We shall argue that $OX-QX$ is less than the radius of the circle. Thus all points X lie on the same side of the hyperbola and therefore Ba is a tangent. To make this argument note that the triangles OBX and ABX are congruent and hence $OX=AX$. Now QX plus the radius is greater than AX by the

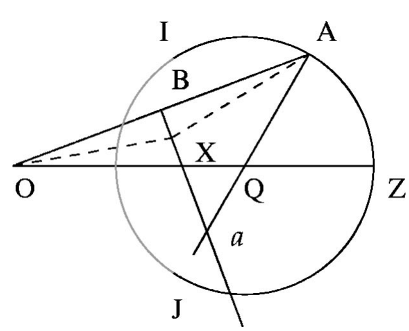

Fig. 9. Same as Fig. 7. X is an arbitrary point on Ba.

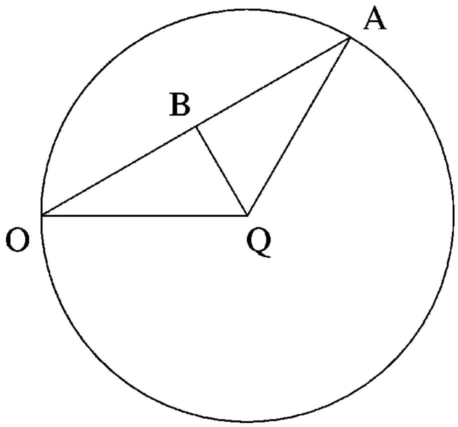

Fig. 10. Failure of the construction of Fig. 4.

triangle inequality applied to the triangle QXA. If we combine these facts, we conclude that the radius is greater than $OX - QX$, the desired inequality.

IV. PARABOLIC MOTION

We first show the failure of the construction of Fig. 4 for the case that the origin O lies on the circumference. Again consider a circle with center Q and O a point on its circumference (Fig. 10). For every point A on the circumference, its image a under the perpendicular bisector construction lies at the same point Q because OQA is now an isosceles triangle; for such a triangle the perpendicular bisector of the base intersects the apex. Thus Feynman's construction degenerates for the case of parabolic motion.

We now describe our modification. Figure 11 shows a circle with center Q. O and Z are diametrically opposite points on the circumference and A is an arbitrary point on the circumference. Draw a tangent to the circle through Z and extend the line OA to it. Through the point of intersection B we draw a line parallel to OZ. Its intersection with QA extended is a, the image of A under this construction. As A is moved around the circle, its image traces the dotted curve shown in Fig. 11. We claim that this curve is a parabola with Q its focus. Moreover, the perpendicular bisector of AB is the tangent to this parabola at a.

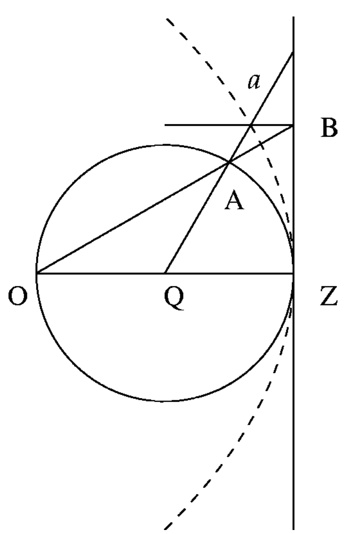

Fig. 11. A circle with center Q. O and Z are diametrically opposite points on the circumference. BZ is a tangent to the circle and Ba is parallel to OZ.

If these claims are true, the dotted curve has the shape of the planetary orbit. We can see this as before by moving the sun to position Q and regarding the circle as the planetary hodograph with O the origin in velocity space. It then follows that the dotted curve has its tangent correctly oriented all along its length.

To prove that the dotted curve is a parabola, note that OQA is an isosceles triangle because $QO = QA$. Hence $\angle QOA = \angle QAO$. Moreover $\angle aAB = \angle QAO$ and $\angle aBA = \angle QOA$ (recall that aB and OQ are parallel). Thus aAB is also an isosceles triangle and $Aa = aB$. Finally, $Qa - aB = aQ - aA$ equals the radius of the circle. aB is perpendicular to BZ because it is parallel to OZ and every tangent to a circle is perpendicular to the corresponding diameter. Thus we have shown that for every point on the dotted curve, its distance from Q minus its distance to the line BZ has the same value (equal to the radius of the circle). Thus by definition the dotted curve is a parabola with Q as its focus.

The tangent at a is the line that bisects the angle AaB. A plausibility argument for this assertion is the following. A light ray from the focus Q to a would be reflected by the angle bisector to ∞ along the line Ba, precisely the behavior we would expect from a parabolic mirror. A full proof that the angle bisector is the tangent is given in the Appendix. Now because AaB is an isosceles triangle, it follows that the bisector of the angle AaB is the perpendicular bisector of AB. Thus the perpendicular bisector of AB is indeed the tangent at a to the parabola.

V. DISCUSSION

The chief virtue of Feynman's method is that it offers another perspective on Kepler's simple and important law of ellipses. It thus complements the analytical method customarily taught in classical mechanics. However, it cannot be said that Feynman's method is purely geometric or that it uses geometry to bypass calculus. Calculus is implicitly invoked when we speak of the tangent to a curve and again when it is asserted that if two curves have parallel tangents at corresponding points, they must have the same shape. Limiting processes are also used to argue that the orbit in velocity space is a circle. Feynman makes this point by remarking that his derivation is "essentially geometric" not "purely geometric...because I don't know what that means."[4] Although it is different in detail, the same remark must apply to Newton's method. We cannot expect to divine Newton's motives in developing his arguments geometrically, but it is doubtful that it was in a quest for greater rigor or to bypass calculus.[8]

ACKNOWLEDGMENTS

It is a pleasure to acknowledge stimulating conversations with Craig Copi, Lawrence Krauss, and Tanmay Vachaspati.

APPENDIX

Figure 12(a) shows a point P that lies on a parabola with a given focus and baseline. Thus the distance of P from the focus minus its distance from the baseline is a constant l that

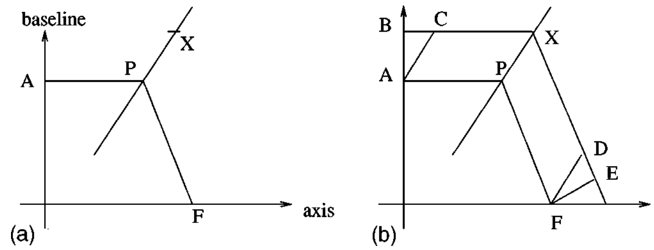

Fig. 12. (a) P is a point on a parabola with focus F and the baseline shown. PA is perpendicular to the baseline and X is an arbitrary point on the line that bisects the angle APF. (b) XB is parallel to AP, XE parallel to PF, AC and FD parallel to PX, and FE is perpendicular to XE.

characterizes the parabola. We wish to show that the line that bisects the angle APF is tangent to the parabola at P. To this end we will show that for any point X on this line, its distance from the focus F minus its distance from the baseline is greater than l. Hence, this line lies entirely to one side of the parabola and is therefore tangent to it.

To make this argument it is helpful to first draw XB parallel to AP, XE parallel to PF, AC and FD parallel to PX, and FE perpendicular to XE. We wish to compare $FX-BX$ with $FP-AP$. First observe that the triangles DEF and CBA are congruent because $\angle FDE=\angle ACB$, $\angle FED=\angle ABC=90°$, and $AC=FD$. Thus $ED=BC$. Next observe that

$$\begin{aligned} EX-BX &= (ED+DX)-(XC+BC) \\ &= DX-XC=FP-AP. \end{aligned} \tag{A1}$$

Moreover $FX>EX$, because FX is the hypotenuse of the right triangle FXE. Thus $FX-BX>EX-BX$. Together with Eq. (A1) this inequality implies that $FX-BX>FP-AP$, which was to be shown.

[a]Electronic mail: hxm7@po.cwru.edu

[1]I. Newton, *Principia Mathematica* (London, 1686); Motte's 1729 translation revised by F. Cajori (University of California Press, Berkeley, 1934).
[2]S. Chandrasekhar, *Newton's Principia for the Common Reader* (Oxford U.P., Oxford, 1995).
[3]R. Feynman, *The Character of Physical Law* (MIT, Cambridge, MA, 1967).
[4]D. Goodstein and J. Goodstein, *Feynman's Lost Lecture* (W. W. Norton, New York, 1996).
[5]J. C. Maxwell, *Matter and Motion* (Dover, New York, 1991). The Dover edition is a reprint of a 1920 edition published by the Society for Promoting Christian Knowledge (London, 1920) with notes and an appendix by Sir J. Larmor.
[6]Remark preceding Proposition XII, Sec. III, Book I of Ref. 1.
[7]Classical mechanics aficionados will recall that the distance between the center and the origin is determined by the Runge-Lenz vector; See, for example, H. Goldstein, *Classical Mechanics* (Addison–Wesley, Reading, MA, 1980), 2nd ed., Chap. 4, p. 125, Problem 24. Needless to say, we shall make no use of such higher dynamics in our elementary discussion.
[8]For interesting commentary on this point, see for example, p. 273 of Ref. 2; R. Westfall, *Never at Rest* (Cambridge University Press, Cambridge, 1980), particularly the discussion in Chap. 10, pp. 423–425 of the paperback edition; and J. E. Littlewood, *Littlewood's Miscellany* (Cambridge U.P., Cambridge, 1986). The essence of the argument is to show that if an object moves away from the sun it must slow down. It is sufficient to show this for an infinitesimal motion.