Feynman's propagator applied to network models of localization

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Network models of dirty electronic systems are mapped onto an interacting field theory of lower dimensionality by interpreting one space dimension as time. This is accomplished via Feynman's interpretation of antiparticles as particles moving backwards in time. The method developed maps calculation of the moments of the Landauer conductance onto calculation of correlation functions of an interacting field theory of bosons and fermions. The resulting field theories are supersymmetric and closely related to the supersymmetric spin-chain representations of network models recently discussed by various authors. As an application of the method, the two-edge Chalker-Coddington model is shown to be Anderson localized, and a delocalization transition in a related two-edge network model (recently discussed by Balents and Fisher) is studied by calculation of the average Landauer conductance. [S0163-1829(97)04047-2]

I. INTRODUCTION

Dirty electronic systems exhibit a variety of interesting phases and transitions in their transport properties at low temperature. For example, consider two-dimensional electrons moving in a random potential and a strong perpendicular field, as in a quantum Hall experiment. Generically the wave functions are localized, with tails that decay over a length scale called the localization length, which leads to insulating behavior at low temperature. However, if the parameters (e.g., the magnetic field or electron density) are tuned to special isolated values the localization length diverges. Such delocalization transitions were postulated shortly after the discovery of the quantum Hall effect and invoked in order to explain it.[1] Since then impressive strides have been made in experimental characterization of these transitions;[2] but a satisfactory theoretical understanding has not yet emerged.[3]

An important theoretical advance was made by Chalker and Coddington who introduced a simplified network model of the quantum Hall transition.[4] Numerical studies show this model produces the same universal behavior at the delocalization transition as more literal (and more complicated) models of the quantum Hall system.[3] Because the network model is relatively simple and is based on a clear physical picture of the transition, it seems a promising starting point for a controlled approximate analysis of the transition.

More recently it has been observed that the random bond Ising model is closely related to a variation on the network model.[5] Progress in analysis of network models is therefore desirable from this point of view also.

Conductance is a sensitive probe of delocalization. The purpose of this paper is to introduce a technique suitable for calculation of the conductance of network models. Following Landauer we imagine electrons are injected from the source into the sample where they undergo multiple scattering.[6] Eventually an electron may either get scattered forward into the drain or it may get scattered backward into the source (see Fig. 1). We wish to calculate the probability of forward scattering.

It is often fruitful in statistical mechanics to map a problem onto a quantum field theory of lower dimensionality by reinterpreting one space dimension as time. In applying that strategy here it becomes neccessary to take into account the fact that electrons will then appear to move both forwards and backwards in ''time.''

A similar situation is encountered in quantum electrodynamics: in a famous paper, Feynman showed that it is possible to regard positrons as electrons moving backwards in time.[7] Following Feynman it is possible here to interpret the x axis in Fig. 1 as time provided we regard an electron moving to the left as a sort of antiparticle of an electron moving to the right. For example, the process in which a right moving electron is scattered to the left could be regarded as a process in which the particle meets its antiparticle leading to their mutual annihilation. From this perspective, the simple process depicted in Fig. 1 where an electron zigzags through the sample could be regarded as a process in which a particle-antiparticle pair is created (at the earlier kink in the trajectory) and annihilated (at the later kink).

Based on this interpretation, it is possible to map a two-dimensional network model onto a one-dimensional field theory of particles and antiparticles. The statistics of the particles may be taken to be either Bose or Fermi. For technical reasons it is most convenient to introduce both species. Thus calculation of the conductance is mapped onto calculation of

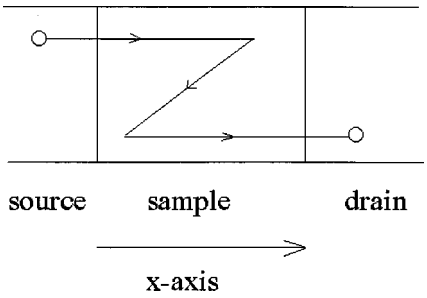

FIG. 1. Landauer's picture: The electron undergoes multiple scattering in the sample and is eventually scattered forward into the drain. In the field theory representation this corresponds to a process involving pair creation and annihilation.

the correlation functions of an interacting field theory of bosons and fermions.

For clarity we illustrate the method on two-edge network models, which are essentially one dimensional. They map onto quantum mechanics problems of interacting bosons and fermions—a zero-dimensional field theory. Generalization of the mapping to a two-dimensional network model is straightforward. Indeed the method should be much more broadly applicable to dirty electronic systems. Since an important obstacle to nonperturbative analysis of random systems has been the lack of suitable representations of the problem, it is hoped this method may prove useful. The present method is closely related to supersymmetric spin-chain representations of network models that have recently been discussed by several authors.[8–13]

As an application of the method we analyze two different network models. The first is essentially a one-dimensional Chalker-Coddington model that exhibits Anderson localization: for a large enough sample the zero-temperature conductance is found to decay exponentially with sample size.

The second model was recently discussed in the context of quantum transport by Balents and Fisher.[13] It is related to a model of glass first studied by Dyson[14] and to a special version of the random bond Ising model introduced by McCoy and Wu.[15] An enlightening discussion of these connections, with references, is given in the paper of Balents and Fisher. This model is known to have a critical point at a special value of its parameters that is of considerable interest as a simple example of a random quantum critical point. Much is known about this model and its relations, particularly through the application of real-space renormalization-group methods by Fisher.[16] In their recent paper Balents and Fisher have applied supersymmetry methods to calculate the exact two-parameter scaling function of the Green's function of this model. Here we shall study its delocalization transition by calculating the conductance. A summary of our results for this model is given in Sec. IV C.

II. FIELD THEORY FORMULATION

In this section a simple two-edge network model is introduced and its Landauer conductance defined. The main result is an exact formal expression for the Landauer conductance, Eqs. (42)–(44), within a field theory formulation of the problem.

A. Landauer conductance

The models considered in this paper consist of two counterpropagating ''edge states'' coupled by tunneling. Along each one-dimensional edge electrons can propagate in only one direction (see Fig. 2). The electron wave function has two components: $\psi_+(x)$, the amplitude to be on the right moving edge at position x, and $\psi_-(x)$, the amplitude to be on the left moving edge at position x. The time-independent Schrödinger equation governing this model is

$$\begin{pmatrix} -i\dfrac{\partial}{\partial x} & m(x) \\ m^*(x) & i\dfrac{\partial}{\partial x} \end{pmatrix} \begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix} = E \begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix}. \tag{1}$$

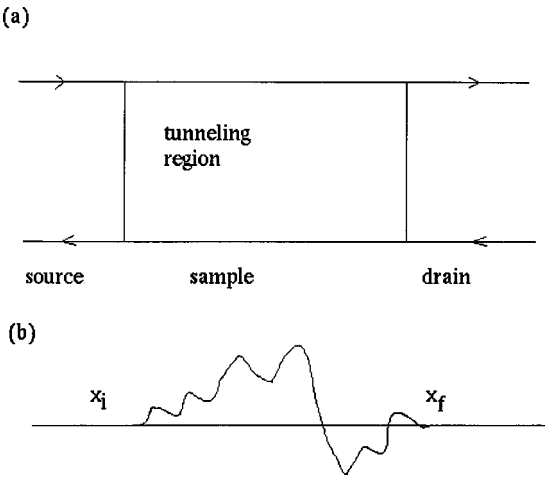

FIG. 2. (a) A two-edge network model. (b) Schematic picture of the random function $m(x)$.

The tunneling amplitude $m(x)$ is some given function. Eventually we will take tunneling to be a random process and will be interested in performing averages over an ensemble of different realizations of $m(x)$ with statistics to be described below. We are interested in solutions at some fixed energy E (the Fermi energy).

For later reference, it is useful to rewrite Eq. (1) in the form

$$-i\frac{\partial}{\partial x}\begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix} = \begin{pmatrix} E & -m(x) \\ m^*(x) & -E \end{pmatrix} \begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix}. \tag{2}$$

An obvious generalization of this model is to consider $2N$ counterpropagating edge states. With suitable statistical assumptions about the tunneling these generalizations lead to the Chalker-Coddington model of the quantum Hall transition[4] or the random bond Ising model.[5] The model studied here can be considered a special anisotropic case in which tunneling between alternate pairs of edges has been turned off.

Following Landauer's method,[6] the model is separated into the sample (region between x_i and x_f) and the probes (regions to the left of x_i and to the right of x_f); see Fig. 2. The left probe is called the source; the right probe, the drain. In contrast to the sample the probes are assumed to be disorder free. $m(x)$ is therefore random only for $x_i < x < x_f$. It is constant in the probes. For simplicity we shall assume that the edges are uncoupled in the probes [that is, $m(x)=0$].[18] This is shown schematically in Fig. 2(b).

The wave equation (2) is now trivially soluble in the probes. We find

$$\begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix} = \begin{pmatrix} \alpha \exp iEx \\ \beta \exp -iEx \end{pmatrix} \quad \text{(source)}$$

$$= \begin{pmatrix} \gamma \exp iEx \\ \delta \exp -iEx \end{pmatrix} \quad \text{(drain)}. \tag{3}$$

For a given realization of $m(x)$ the connection between (γ,δ) and (α,β) can be found by integrating Eq. (2) across the sample:

$$\begin{pmatrix} \gamma \\ \delta \end{pmatrix} = T \begin{pmatrix} \alpha \\ \beta \end{pmatrix}. \tag{4}$$

T is a 2×2 matrix called the transfer matrix of the sample. The transfer matrix is the focus of numerical studies of network models.[3]

Here we are interested in solving Eq. (2) subject to scattering boundary conditions in the probes so that

$$\alpha \to 1, \quad \beta \to r; \quad \gamma \to t, \quad \delta \to 0. \tag{5}$$

According to Landauer the zero-temperature conductance of the sample (at Fermi energy E) is given by

$$g = \frac{e^2}{h}|t|^2. \tag{6}$$

Returning to Eq. (2) it is useful to regard $x \to$ time and to interpret it as the *time-dependent* Schrödinger equation of a two-level system. An awkward feature of such a reinterpretation is that the 2×2 ''Hamiltonian'' matrix in Eq. (2) is not Hermitean.[19] Sadly this feature will persist through much of our analysis and appears to be more generally pervasive.[8,12,13]

We can now imagine calculating the retarded propagator for Eq. (2). Given the wave function (values of ψ_+ and ψ_-) at a particular x slice, the retarded propagator gives the wave function for ''future'' values of x. For example, if the wave function is known at x_i, the retarded propagator gives the wave function inside the sample (and beyond, in the drain). This is in very much the same spirit as the transfer matrix (indeed the retarded propagator *is* the transfer matrix for a special value of its arguments). It is not helpful for calculating the scattering amplitudes.

While solving Eq. (2) subject to scattering boundary conditions we do not know the complete wave function for any x slice. Instead we know the positive frequency component $\psi_+ = 1$ at the earlier slice x_i and we know the negative frequency component $\psi_- = 0$ at a later slice x_f. We would like to reconstruct the wave function between the slices using this information, or at least we would like to know $\psi_+(x \to x_f) = t$.

Feynman showed precisely that this would be achieved by a modification of the retarded propagator, now known as the Feynman propagator.[7] Feynman derived an integral equation obeyed by his propagator and a perturbation series for it; this was the main focus of his paper. Peripherally, in two Appendices, he showed the same results could be deduced from a second quantized theory of particles and antiparticles. This development will now be used to provide a field theory representation of the two edge network model [Eq. (2)].[20]

B. Deduction from second quantization

The purpose of this section is to derive a second quantized Hamiltonian from which one can calculate the Feynman propagator and scattering amplitude t of Eq. (2).

1. Fermion representation

Return to the interpretation of Eq. (2) as a time-dependent two-level system. Introduce $a_+^\dagger$, which creates the right moving state $\binom{1}{0}$, $a_-^\dagger$, which creates the left moving state $\binom{0}{1}$, and a_+ and a_-, the corresponding destruction operators. The a's obey Fermi (anti-)commutation rules. Also introduce the ''time''-dependent Hamiltonian

$$H(x) = (a_+^\dagger a_-^\dagger) \begin{pmatrix} E & -m(x) \\ m^*(x) & -E \end{pmatrix} \begin{pmatrix} a_+ \\ a_- \end{pmatrix}. \tag{7}$$

This Hamiltonian is not useful for our purpose. It can be shown to generate the retarded propagator.

Instead we must now introduce c^R fermions, related to the a fermions via the particle-hole transformation

$$c_+^{R\dagger} = a_+^\dagger, \quad c_-^{R\dagger} = a_-,$$
$$c_+^R = a_+, \quad c_-^R = a_-^\dagger. \tag{8}$$

The superscript R is superfluous for the moment; its function will become apparent later. In terms of the c^R fermions the Hamiltonian is

$$H_F^R(x) = E(c_+^{R\dagger}c_+^R + c_-^{R\dagger}c_-^R) - m(x)c_+^{R\dagger}c_-^{R\dagger} + m^*(x)c_-^R c_+^R. \tag{9}$$

The scattering amplitude t can be computed straightforwardly using this Hamiltonian; see Eq. (19) below. In the remainder of this subsubsection that result will be derived.

Note that H_F is non-Hermitean—this is traceable to the non-Hermiticity of the ''Hamiltonian'' of Eq. (2). The reader troubled by this non-Hermiticity should again read Ref. 19. Note also that H_F does not conserve the total number of c fermions. Instead we may regard the c_- fermion as the ''antiparticle'' of the c_+. What is conserved then is the total ''charge''—the difference in the number of particles and antiparticles, given by

$$Q_F = c_+^{R\dagger}c_+^R - c_-^{R\dagger}c_-^R. \tag{10}$$

The S matrix corresponding to H_F^R is the solution to

$$-i\frac{\partial}{\partial x}S_F^R(x,x_i) = H_F^R(x)S_F^R(x,x_i), \tag{11}$$

subject to the initial condition $S_F^R(x_i,x_i) = 1$.

Turning to the derivation of Eq. (19), we first obtain a useful formula, following Feynman. Let $[e_+(x), e_-(x)]$ be a solution to the wave equation (2). Construct the operator $\hat{F} \equiv e_+(x_i)c_+^{R\dagger} + e_-(x_i)c_-^R$. $\hat{F}$ is transformed by the S matrix as follows:

$$S_F^R(x,x_i)\hat{F}S_F^{R-1}(x,x_i) = e_+(x)c_+^{R\dagger} + e_-(x)c_-^R. \tag{12}$$

This is Feynman's formula. To prove it, regard Eq. (12) as an ansatz; in other words assume that $S\hat{F}S^{-1}$ is of this form with $e_+(x)$ and $e_-(x)$ some suitable functions. Upon differentiation with respect to x, it will be seen that the ansatz is consistent provided (e_+, e_-) obey the wave equation (2).

Carrying out this plan, we note that

$$-i\frac{\partial}{\partial x}S_F^{R-1}(x,x_i) = -S_F^{R-1}(x,x_i)H_F^R(x) \tag{13}$$

obtained by differentiating $S_F^{R-1}S_F^R=1$ and using the evolution Eq. (11) for the S matrix. Equations (11), (12), and (13) together yield

$$-i\frac{\partial}{\partial x}(S_F^R\hat{F}S_F^{R-1})=[H_F,S_F^R\hat{F}S_F^{R-1}]$$
$$=e_+(x)[H_F,c_+^{R\dagger}]+e_-(x)[H_F,c_-^R]. \tag{14}$$

The relevant commutators are

$$[H_F,c_+^{R\dagger}]=Ec_+^{R\dagger}+m^*c_-^R,$$
$$[H_F,c_-^R]=-Ec_-^R-mc_+^{R\dagger}. \tag{15}$$

Hence the derivative of the left side of the ansatz Eq. (12) is

$$-i\frac{\partial}{\partial x}(S_F^R\hat{F}S_F^{R-1})=c_+^{R\dagger}[Ee_+(x)-m(x)e_-(x)]$$
$$+c_-^R[m^*(x)e_+(x)-Ee_-(x)]. \tag{16}$$

On comparing with the corresponding x derivative of the right hand side of the ansatz it is seen that indeed (e_+,e_-) must obey the wave equation (2); this completes the proof of Feynman's formula, Eq. (12).

Next let (e_+,e_-) be the special solution that obeys scattering boundary conditions:

$$e_+(x_i)=1,\quad e_-(x_i)=r,$$
$$e_+(x_f)=t,\quad e_-(x_f)=0. \tag{17}$$

This leads to the *scattering* formula

$$S_F^R(x_f,x_i)(c_+^{R\dagger}+rc_-^R)S_F^{R-1}(x_f,x_i)$$
$$=tc_+^{R\dagger}\Rightarrow S_F^R(x_f.x_i)c_+^{R\dagger}$$
$$=tc_+^{R\dagger}S_F^R(x_f,x_i)-rS_F^R(x_f,x_i)c_-^R, \tag{18}$$

a useful special case of Feynman's formula, Eq. (12).

Finally analyze the matrix element $\langle 0|c_+^RS_F^R(x_f,x_i)c_+^{R\dagger}|0\rangle$; here $|0\rangle$ is the vacuum for the c fermions. Use of the scattering formula, Eq. (18), reveals

$$t=\frac{\langle 0|c_+^RS_F^R(x_f,x_i)c_+^{R\dagger}|0\rangle}{\langle 0|S_F^R(x_f,x_i)|0\rangle}. \tag{19}$$

Equation (19) shows that the scattering amplitude t can be computed by studying the evolution of the single-particle state $c_+^{R\dagger}|0\rangle$ under the Hamiltonian of Eq. (9). Note that the vacuum amplitude in the denominator of Eq. (19) is not trivial. In Feynman's words, ''It differs from unity because, for example, a pair could be created which eventually annihilates itself.''

Equation (19) is the main result of this subsubsection. By itself, Eq. (19) is not especially helpful. Our eventual purpose is to compute averages over the random tunneling process $m(x)$. Since $m(x)$ appears in both numerator and denominator of Eq. (19), this form is not particularly well adapted for averaging.

2. Boson representation

Tracing through the derivation of Eq. (19) it is seen that the Fermi statistics of the c^R particles plays no crucial role; only the commutators of Eq. (15) are essential. Hence we may replace the c^R fermions with b^R bosons governed by the Hamiltonian

$$H_B^R(x)=E(b_+^{R\dagger}b_+^R+b_-^{R\dagger}b_-^R)+m(x)b_+^{R\dagger}b_-^{R\dagger}+m^*(x)b_-^Rb_+^R. \tag{20}$$

Apart from the replacement $c^R\rightarrow b^R$, this differs from Eq. (9) in the sign of the pair creation term (the term proportional to $b_+^{R\dagger}b_-^{R\dagger}$). This sign change ensures commutation relations of the desired form similar to Eq. (15):

$$[H_B^R,b_+^{R\dagger}]=Eb_+^{R\dagger}+m^*(x)b_-^R,$$
$$[H_B^R,b_-^{R\dagger}]=-Eb_-^R-m(x)b_+^{R\dagger}. \tag{21}$$

Note that due to the sign change in the pair creation term the boson Hamiltonian, Eq. (20), is actually Hermitean. Again the b_- boson may be regarded as the antiparticle of the b_+. The Hamiltonian conserves the difference in the number of particles and antiparticles, that is, the charge

$$Q_B=b_+^{R\dagger}b_+^R-b_-^{R\dagger}b_-^R. \tag{22}$$

The entire analysis of the preceding subsubsection can now be carried over essentially unchanged. The S matrix corresponding to H_B^R is the solution to

$$-i\frac{\partial}{\partial x}S_B^R(x,x_i)=H_B^R(x)S_B^R(x,x_i) \tag{23}$$

subject to the initial condition $S_B^R(x_i,x_i)=1$. Feynman's formula, Eq. (12), and the scattering formula, Eq. (18), apply upon making the replacement $c^R\rightarrow b^R$. Using the scattering formula we can deduce

$$t=\frac{\langle 0|b_+^RS_B^R(x_f,x_i)b_+^{R\dagger}|0\rangle}{\langle 0|S_B^R(x_f,x_i)|0\rangle}, \tag{24}$$

the bosonic analog of Eq. (19).

C. Analysis of the vacuum problem

Remarkably, the vacuum amplitudes for the bosons and fermions cancel; that is,

$$\langle 0|S_F^R(x_f,x_i)|0\rangle\langle 0|S_B^R(x_f,x_i)|0\rangle=1. \tag{25}$$

We have previously derived a fermionic and bosonic expression for the scattering amplitude, neither well suited for performing an average over the random tunneling process, $m(x)$. Equation (25) will enable us to weld these expressions into a form suitable for averaging. We return to this point in Sec. II E below. Here we focus on proving Eq. (25).

To calculate the vacuum amplitude, following Feynman, let us analyze a series of problems that interpolate smoothly between a soluble limit and the problem we want to solve. Introduce a truncated problem for which $m(x)$ is left unchanged for $x_0<x<x_f$ and is set equal to zero for $x_i<x<x_0$. By varying x_0 we obtain the desired series of problems. Evi-

dently $x_0=x_i$ is the case we wish to solve. On the other hand, if $x_0=x_f$, there is no tunneling and the problem is trivially soluble.

Denote the scattering coefficients for the truncated problem $r(x_0)$ and $t(x_0)$. For the soluble case, $r(x_f)=0$ and $t(x_f)=1$.

The fermion S matrix for the truncated problem will be written as $S^{RF}_{x_0}(x,x_0)$. It obeys

$$-i\frac{\partial}{\partial x}S^{RF}_{x_0}(x,x_0)=H^R_F(x)S^{RF}_{x_0}(x,x_0), \tag{26}$$

with $S^{RF}_{x_0}(x_0,x_0)=1$. The truncated boson S matrix is similarly defined.

The vacuum amplitude for the truncated fermion problem will be denoted $C^R_F(x_0)$. Thus

$$C^R_F(x_0)=\langle 0|S^{RF}_{x_0}(x_f,x_0)|0\rangle. \tag{27}$$

Analogous expressions can be written for the bosonic case. When $x_0=x_f$ the vacuum amplitude is unity for both bosons and fermions.

The strategy now is to study the variation of $C(x_0)$ with x_0 by writing a differential equation for it. A formal solution of the differential equation can be obtained that will suffice to prove the cancellation of the vacuum amplitude.

For this purpose it is useful to write the solution to Eq. (26) as a formal series

$$S^{RF}_{x_0}(x_f,x_0)=1+i\int_{x_0}^{x_f}dx_1H^R_F(x_1)+i^2\int_{x_0}^{x_f}dx_1\int_{x_0}^{x_1}dx_2H^R_F(x_1)H^R_F(x_2)+\cdots. \tag{28}$$

By differentiation of this series it follows that

$$-i\frac{\partial}{\partial x_0}S^{RF}_{x_0}(x_f,x_0)=-S^{RF}_{x_0}(x_f,x_0)H^R_F(x_0). \tag{29}$$

Taking the vacuum expectation of Eq. (29) we obtain

$$-i\frac{\partial}{\partial x_0}C^{RF}_{x_0}=-\langle 0|S^{RF}_{x_0}(x_f,x_0)H^R_F(x_0)|0\rangle;$$

$$H^R_F(x_0)=-m(x_0)c^{R\dagger}_+c^{R\dagger}_-+\text{others}. \tag{30}$$

In Eq. (30) we have explicitly written only that term of H^R_F [Eq. (9)] that does not annhilate the vacuum.

According to the scattering formula, Eq. (18),

$$S^{RF}_{x_0}(x_f,x_0)c^{R\dagger}_+=t(x_0)c^{R\dagger}_+S^{RF}_{x_0}(x_f,x_0)-r(x_0)S^{RF}_{x_0}(x_f,x_0)c^R_-. \tag{31}$$

Applying this to the matrix element in Eq. (30) yields the differential equation obeyed by $C^R_F(x_0)$:

$$-i\frac{\partial}{\partial x_0}C^R_F(x_0)=-m(x_0)r(x_0)C^R_F(x_0). \tag{32}$$

We seek a solution subject to $C^R_F(x_f)=1$.

Define $L(x_0)$ as the solution of

$$-i\frac{\partial}{\partial x_0}L(x_0)=-m(x_0)r(x_0) \tag{33}$$

with the initial condition $L(x_f)=0$. Given $m(x)$ one can imagine solving the truncated problem for various values of x_0 and, in principle, computing $L(x_0)$. The solution of Eq. (32) is then

$$C^R_F(x_0)=\exp L(x_0). \tag{34}$$

Note that this solution obeys the differential equation (32) *and* the initial condition $C^R_F(x_f)=1$.

The same analysis leads to the boson vacuum amplitude

$$C^R_B(x_0)=\exp -L(x_0). \tag{35}$$

The sign change is traceable to the sign difference in the pair creation terms of the Fermi and Bose Hamiltonians. Hence $C^R_F(x_0)C^R_B(x_0)=1$ for all x_0 and in particular for $x_0=x_i$, which establishes the desired result, Eq. (25).

It is interesting that Feynman appears to have devoted some effort to interpreting Dirac particles as bosons even though it violated the spin-statistics theorem. In particular, the result of Eq. (25), which is of no obvious value in quantum electrodynamics, is derived by Feynman.

D. Conjugate amplitude

The Landauer conductance is given by $|t|^2$. Thus expressions for t^* analogous to Eqs. (19) and (24) will be needed. To obtain them consider the equation conjugate to Eq. (2):

$$-i\frac{\partial}{\partial x}\begin{pmatrix}\phi_+\\ \phi_-\end{pmatrix}=\begin{pmatrix}-E & m^*(x)\\ -m(x) & E\end{pmatrix}\begin{pmatrix}\phi_+\\ \phi_-\end{pmatrix}. \tag{36}$$

Equation (36) is constructed to have the property that if (e_+,e_-) is a solution to Eq. (2), then the complex conjugate (e^*_+,e^*_-) is a solution to Eq. (36). Hence if Eq. (36) is solved subject to scattering boundary conditions, the scattering amplitude will be t^*.

On the other hand, direct comparison reveals that Eq. (2) is transformed into Eq. (36) by the replacements $E\to -E, m\to -m^*, m^*\to -m$. Hence introduce c^A fermions governed by the Hamiltonian

$$H^A_F(x)=-E(c^{A\dagger}_+c^A_++c^{A\dagger}_-c^A_-)+m^*(x)c^{A\dagger}_+c^{A\dagger}_--m(x)c^A_-c^A_+. \tag{37}$$

This Hamiltonian is obtained from Eq. (9) by making the replacements indicated above. By the reasoning that lead from Eq. (9) to (19), H^A_F generates the conjugate amplitude via

$$t^*=\frac{\langle 0|c^A_+S^A_F(x_f,x_i)c^{A\dagger}_+|0\rangle}{\langle 0|S^A_F(x_f,x_i)|0\rangle}, \tag{38}$$

where $S^A_F(x_f,x_i)$ is the S matrix corresponding to H^A_F.

Similarly, one can write

$$t^*=\frac{\langle 0|b^A_+S^A_B(x_f,x_i)b^{A\dagger}_+|0\rangle}{\langle 0|S^A_B(x_f,x_i)|0\rangle}, \tag{39}$$

where $S^A_B(x_f,x_i)$ is the S matrix of the boson Hamiltonian

$$H_B^A = -E(b_+^{A\dagger}b_+^A + b_-^{A\dagger}b_-^A) - m^*(x)b_+^{A\dagger}b_-^{A\dagger} - m(x)b_-^A b_+^A . \tag{40}$$

Finally note that the vacuum amplitudes for the A bosons and fermions cancel, as they do for their R counterparts:

$$\langle 0|S_F^A(x_f,x_i)|0\rangle\langle 0|S_B^A(x_f,x_i)|0\rangle = 1. \tag{41}$$

E. Supersymmetry

The results of the previous subsections can now be assembled into an expression for the Landauer conductance suitable for averaging over the random tunneling process $m(x)$. Simultaneously introduce c^R and c^A fermions and b^R and b^A bosons governed by the total Hamiltonian

$$H_{\rm SUSY}(x) = H_F^R + H_F^A + H_B^R + H_B^A = E\hat{M} + m(x)\hat{A} + m^*(x)\hat{B}. \tag{42}$$

Here $\hat{M} \equiv (c_+^{R\dagger}c_+^R + c_-^{R\dagger}c_-^R - R \rightarrow A) + (b_+^{R\dagger}b_+^R + b_-^{R\dagger}b_-^R - R \rightarrow A)$ and $\hat{A} \equiv -c_+^{R\dagger}c_-^{R\dagger} - c_-^A c_+^A + b_+^{R\dagger}b_-^{R\dagger} - b_-^A b_+^A$ and $\hat{B} \equiv c_-^R c_+^R + c_+^{A\dagger}c_-^{A\dagger} + b_-^R b_+^R - b_+^{A\dagger}b_-^{A\dagger}$.

As usual the corresponding S matrix obeys

$$-i\frac{\partial}{\partial x}S_{\rm SUSY}(x,x_i) = H_{\rm SUSY}(x)S_{\rm SUSY}(x,x_i) \tag{43}$$

and the initial condition $S_{\rm SUSY}(x_i,x_i) = 1$.

The Landauer conductance can be calculated using

$$|t|^2 = \langle 0|c_+^A c_+^R S_{\rm SUSY}(x_f,x_i)c_+^{R\dagger}c_+^{A\dagger}|0\rangle. \tag{44}$$

To see this note that since the different particle species do not interact, $S_{\rm SUSY} = S_F^R S_F^A S_B^R S_B^A$; hence the matrix element in Eq. (44) decouples into

$$\langle 0|C_+^R S_F^R c_+^{R\dagger}|0\rangle\langle 0|c_+^A S_F^A c_+^{A\dagger}|0\rangle\langle 0|S_B^R|0\rangle\langle 0|S_B^A|0\rangle.$$

Using Eqs. (19), (38), (25), and (41), this product is easily seen to be $|t|^2$.

Equation (44) is our principal tool to analyze the network model. It shows that the conductance can be calculated by following the evolution of a two fermion state under the Hamiltonian $H_{\rm SUSY}$. The principal advantage of this expression is due to the absence of a denominator, which makes it well suited for averaging over the random tunneling process $m(x)$. The averaging will be discussed further in Secs. III and IV assuming different distributions for $m(x)$. The reader has perhaps noticed that one could develop various other expressions for the Landauer conductance that involve evolving, instead of a two-fermion state, a two-boson state or a one-boson, one-fermion state; but Eq. (44) is the form we shall use here.

Finally let us briefly examine some symmetries of $H_{\rm SUSY}$. Evidently, it conserves the R fermion charge; that is,

$$[Q_F^R, H_{\rm SUSY}] = 0, \tag{45}$$

where $Q_F^R \equiv c_+^{R\dagger}c_+^R - c_-^{R\dagger}c_-^R$. Similarly it also conserves the R boson charge as well as the corresponding quantity for the A boson and fermion. $H_{\rm SUSY}$ also possesses a supersymmetry analogous to that displayed by the models studied in Refs. 8–13. Specifically the Hamiltonian, Eq. (42), commutes with the supercharges Q_{FB}^R, Q_{FB}^A, Q_{BF}^R, and Q_{BF}^A which are given by

$$Q_{FB}^R \equiv c_+^R b_+^{R\dagger} + c_-^{R\dagger}b_-^R ,$$

$$Q_{BF}^R \equiv c_+^{R\dagger}b_+^R - c_-^R b_-^{R\dagger} ; \tag{46}$$

Q_{FB}^A and Q_{BF}^A are obtained by replacing $R \rightarrow A$ in Eq. (46).

It will be seen below that after disorder averaging we must analyze an effective interacting Hamiltonian instead of the noninteracting (but random) Hamiltonian of Eq. (42). The effective interacting Hamiltonian will also possess the symmetries discussed above.

III. TWO-EDGE CHALKER-CODDINGTON MODEL

A. Disorder average

First let us assume that the real and imaginary parts of $m(x)$ are independent Gaussian white noise processes with zero mean. Thus $[m(x)]_{\rm ens} = 0$. Here $[\cdots]_{\rm ens}$ denotes an average over the ensemble of $m(x)$ realizations. The variance is given by

$$[m^*(x), m(x')]_{\rm ens} = D\,\delta(x-x'),$$

$$[m(x), m(x')]_{\rm ens} = 0. \tag{47}$$

Essentially the same statistics result if it is assumed that $m(x)$ is a rapidly fluctuating phase factor. These assumptions are similar to those made in the Chalker-Coddington model. For simplicity we suppose $E = 0$. It is shown in Appendix A that this entails no loss of generality or modification of behavior; but the averaging is simpler to describe.

To calculate the average Landauer conductance, we must average the S matrix $S_{\rm SUSY}(x_f,x_i)$ over the ensemble described above. This is accomplished by expanding the S matrix in a formal series [Eq. (28)] and averaging term by term. The result is

$$[S_{\rm SUSY}(x_f,x_i)]_{\rm ens} = \exp[-(D/2)H_{CC}(x_f - x_i)], \tag{48}$$

where

$$H_{CC} \equiv \hat{A}\hat{B} + \hat{B}\hat{A}. \tag{49}$$

$\hat{A}$ and $\hat{B}$ are defined below Eq. (42). The ensemble averaged conductance is then given by

$$[g]_{\rm ens} = \frac{e^2}{h}\langle 0|c_+^A c_+^R \exp\left(-\frac{D}{2}H_{CC}(x_f - x_i)\right)c_+^{R\dagger}c_+^{A\dagger}|0\rangle. \tag{50}$$

Thus in order to calculate the average conductance we need to study the evolution of a two-fermion state under the effective Hamiltonian H_{CC}. In contrast to $H_{\rm SUSY}$, H_{CC} is not random; but the price paid is that it is interacting.

B. Anderson localization

The evaluation of Eq. (50) is simplified by the observation that all of the fermionic bilinears that appear in H_{CC} (namely, $c_+^{R\dagger}c_-^{R\dagger}$, $c_-^R c_+^R$, $c_+^{A\dagger}c_-^{A\dagger}$, and $c_-^A c_+^A$) annihilate the

two-fermion state. Hence the expression for the average conductance, Eq. (50), simplifies to

$$[g]_{\rm ens}=\frac{e^2}{h}\langle 0|_B \exp\left(-\frac{D}{2}H_{CC}^{\rm boson}(x_f-x_i)\right)|0\rangle_B . \quad (51)$$

Here $H_{CC}^{\rm boson}$ is the purely bosonic part of H_{CC} (written explicitly below) and $|0\rangle_B$ is the boson vacuum. Physically then we need only calculate the boson vacuum amplitude.

Now let us analyze the boson Hamiltonian

$$H_{CC}^{\rm boson}=(b_+^{R\dagger}b_-^{R\dagger}-b_-^A b_+^A)(b_-^R b_+^R-b_+^{A\dagger}b_-^{A\dagger}) +(b_-^R b_+^R-b_+^{A\dagger}b_-^{A\dagger})(b_+^{R\dagger}b_-^{R\dagger}-b_-^A b_+^A). \quad (52)$$

In contrast to the full Hamiltonian H_{CC}, the bosonic part is Hermitean. Note that it is also positive definite since it is of the form $\hat{D}^\dagger\hat{D}+\hat{D}\hat{D}^\dagger$ where $\hat{D}\equiv(b_-^R b_+^R-b_+^{A\dagger}b_-^{A\dagger})$. After some manipulation we obtain the more revealing form

$$H_{CC}^{\rm boson}=h_n+h_+ +h_- ,$$

with

$$h_n\equiv 2n_+^R n_-^R+2n_+^A n_-^A+n_+^R+n_-^R+n_+^A+n_-^A+2,$$

$$h_+\equiv -2b_+^{R\dagger}b_-^{R\dagger}b_+^{A\dagger}b_-^{A\dagger},$$

$$h_-\equiv -2b_+^R b_-^R b_+^A b_-^A . \quad (53)$$

Here $n_+^R=b_+^{R\dagger}b_+^R$, etc. Although lengthy, Eq. (53) has a simple content. Consider the bosonic state $|n\rangle_B$, $n=0,1,2,\ldots$. $|n\rangle_B$ is defined as a normalized state that contains n bosons of each kind ($R+$, $R-$, $A+$, and $A-$). $n=0$ corresponds to the boson vacuum. Inspection of Eq. (53) shows that these states are closed under the action of $H_{CC}^{\rm boson}$. Hence we need only consider the block of $H_{CC}^{\rm boson}$ that lies within the invariant subspace spanned by these states.

Our plan therefore is to find the eigenstates of $H_{CC}^{\rm boson}$ that lie within the subspace spanned by $|n\rangle_B$. Expansion of the boson vacuum in terms of these eigenstates will then allow straightforward evaluation of the boson vacuum amplitude and the average Landauer conductance, Eq. (51). Note that the Hermiticity of $H_{CC}^{\rm boson}$ ensures that its eigenstates form a complete set and it is therefore appropriate to use them as a basis.

The coupled boson Hamiltonian, $H_{CC}^{\rm boson}$, is solved in Sec. II C below. It is found to have a continuum of eigenstates denoted $|k\rangle$ with eigenvalue $(1+k^2)/2$. Here $k\in[0,\infty]$. The eigenstates are orthogonal and are normalized so that $\langle k|0\rangle_B=1$ and

$$\langle p|k\rangle=\frac{2}{\pi}\cosh^2\left(\frac{\pi k}{2}\right)\frac{1}{k\sinh(\pi k/2)}\,\delta(k-p). \quad (54)$$

It follows from Eq. (54) and the presumed completeness of the eigenfunctions of $H_{CC}^{\rm boson}$ that

$$\frac{\pi}{2}\int_0^\infty dk\,\frac{k\sinh(\pi k/2)}{\cosh^2(\pi k/2)}|k\rangle\langle k|=\mathcal{I}. \quad (55)$$

Here $\mathcal{I}$ denotes the identity matrix in the subspace spanned by $|n\rangle_B$.

Inserting the resolution of the identity Eq. (55) into the conductance formula, Eq. (51), yields

$$[g]_{\rm ens}=\frac{\pi e^2}{2h}\exp\left(-\frac{D}{4}(x_f-x_i)\right)\int_0^\infty dk\,\frac{k\sinh(\pi k/2)}{\cosh^2(\pi k/2)} \times\exp\left(-\frac{Dk^2}{4}(x_f-x_i)\right). \quad (56)$$

In the limit of large sample size we find

$$[g]_{\rm ens}\approx\frac{e^2}{h}\left(\frac{\pi}{D(x_f-x_i)}\right)^{3/2}\exp\left(-\frac{D}{4}(x_f-x_i)\right). \quad (57)$$

Thus the model exhibits Anderson localization: the conductance decays exponentially for sufficiently large sample size as generally expected of a dirty one-dimensional quantum wire. Equation (57) agrees with the result obtained by Ref. 8.

C. Solution of coupled boson Hamiltonian

We wish to solve the Schrödinger equation

$$H_{CC}^{\rm boson}|\psi\rangle=\lambda|\psi\rangle \quad (58)$$

within the subspace spanned by $|n\rangle_B$. Expand the eigenstate as

$$|\psi\rangle=a_0|0\rangle_B+a_1|1\rangle_B+\cdots+a_n|n\rangle_B+\cdots. \quad (59)$$

The effect of $H_{CC}^{\rm boson}$, Eq. (53), on the states $|n\rangle_B$ is easily computed:

$$h_n|n\rangle_B=(4n^2+4n+2)|n\rangle_B,$$

$$h_+|n\rangle_B=-2(n+1)^2|n+1\rangle_B, \quad (60)$$

$$h_+|n\rangle_B=-2n^2|n-1\rangle_B .$$

Equations (59) and (60) together allow us to write the Schrödinger equation as a three-term recurrence relation

$$(4n^2+4n+2)a_n-2n^2a_{n-1}-2(n+1)^2a_{n+1}=\lambda a_n , \quad (61)$$

subject to $a_{-1}=0$. Our goal now is to solve Eq. (61) for different λ and then orthonormalize the solutions. The process of orthonormalization will weed out the disallowed values of λ. Note that from the Hermiticity and positive definiteness of $H_{CC}^{\rm boson}$ mentioned following Eq. (52), we are already assured that the allowed λ must be positive and real.

To solve the recurrence relation we introduce the generating function

$$f(x)=a_0+a_1x+\cdots+a_nx^n+\cdots . \quad (62)$$

From the recurrence relation, Eq. (61), we can easily construct the differential equation obeyed by f:

$$x(x-1)^2\frac{d^2}{dx^2}f+(3x-1)(x-1)\frac{d}{dx}f+\left(x-1+\frac{\lambda}{2}\right)f=0. \quad (63)$$

This equation has three regular singular points at $x=0$, 1, and ∞; it is therefore a Riemann P equation. One solution is analytic at $x=0$ as can be verified by directly substituting the

series, Eq. (62), in the differential equation (63). This is the solution we seek; it generates the solution to Eq. (61).

The solution to a Riemann P equation can always be expressed in terms of hypergeometric functions. Making the standard transformations (see, e.g., Ref. 17, chapter 5) we find the analytic solution is given by

$$f(x)=(1-x)^{\mu}F(\mu+1,\mu+1,1;x);$$

where

$$\mu=-\frac{1}{2}+\frac{\sqrt{1-2\lambda}}{2}. \tag{64}$$

Note that for $\lambda>1/2$, μ becomes complex. Equation (64) therefore points to some change in behavior at $\lambda=1/2$. Below we will find that in fact the allowed eigenvalues are $\lambda>1/2$.

The coefficients a_n can be extracted from $f(x)$ via the contour integral

$$a_n=\oint_C \frac{dx}{2\pi i}\frac{1}{x^{n+1}}f(x). \tag{65}$$

Here C is a contour that encircles the origin but not the branch point $x=1$. This contour integral cannot be expressed in elementary form in general but the large n behavior of a_n can be obtained by using an integral representation for the hypergeometric function (see Appendix B).

Let us focus on the solutions to Eq. (61) with $\lambda>1/2$. Write $\lambda=1/2+k^2/2$ where $k\in[0,\infty]$. The large-n asymptotic behavior is

$$a_n\approx\frac{2}{\pi}\cosh\frac{\pi k}{2}\,\frac{1}{[2k\sinh(\pi k/2)]^{1/2}}\,\frac{1}{\sqrt{n}}\cos\left(\frac{k}{2}\ln n+\phi(k)\right). \tag{66}$$

Here the phase $\phi(k)=\arg[\Gamma(ik)\Gamma(1/2-ik/2)/\Gamma^2(1/2+ik/2)]$. Thus the solutions for $\lambda>1/2$ decay slowly (as $1/\sqrt{n}$) and exhibit weak logarithmic oscillations.

Fortunately, it turns out that this asymptotic behavior is sufficient to orthonormalize the eigenstates. Let b_n be a solution to Eq. (61) with eigenvalue ρ:

$$(4n^2+4n+2)b_n-2n^2b_{n-1}-2(n+1)^2b_{n+1}=\rho b_n. \tag{67}$$

Multiply Eq. (61) by b_n, Eq. (67) by a_n, take the difference, and sum on n. A series of cancellations allows us to express the sum as a surface term[21]

$$(\rho-\lambda)\sum_{n=0}^{N}a_nb_n=2(N+1)^2(a_{N+1}b_N-a_Nb_{N+1}). \tag{68}$$

Thus the asymptotic behavior of a_n is sufficient to evaluate the orthonormalization sum $\lim_{N\to\infty}\Sigma_{n=0}^{N}a_nb_n$. Use of Eqs. (66) and (68) and the delta function representation[22]

$$\lim_{L\to\infty}\frac{1}{k}\sin kL=\pi\delta(k) \tag{69}$$

yields Eq. (54).

We have so far focused on solutions with $\lambda>1/2$. Using similar arguments one can show that solutions with $0<\lambda<1/2$ do not decay fast enough for large n to be orthonormalizable even as continuum eigenfunctions. They are analogous to the exponential negative energy solutions of the free-particle Schrödinger equation in elementary quantum mechanics. In the same spirit, the solutions for $\lambda>1/2$ are analogous to the positive energy continuum of plane-wave solutions.

Having shown that there are no eigenstates with $\lambda<1/2$, we have justified the step from Eq. (54) to Eq. (55). Alternatively one might try to prove the completeness relation, Eq. (55), directly. Forming the matrix element of Eq. (55) between the states $|n\rangle_B$ and $|m\rangle_B$, one should check whether

$$\frac{\pi}{2}\int_0^{\infty}dk\,\frac{k\sinh(\pi k/2)}{\cosh^2(\pi k/2)}a_m(k)a_n(k)=\delta_{nm}. \tag{70}$$

We have verified this analytically (and more extensively numerically) for a few small values of n and m for which the a's can be computed directly from Eq. (61). It may be possible to construct the general proof using Eqs. (64) and (65), but this passes beyond conventional standards of rigor and good taste in theoretical physics.

IV. DELOCALIZATION TRANSITION IN DYSON GLASS

Now let us analyze the case in which $m(x)$ is purely imaginary. We rewrite Eq. (2) as

$$-i\frac{\partial}{\partial x}\begin{pmatrix}\psi_+\\ \psi_-\end{pmatrix}=\begin{pmatrix}E & -im\\ -im & -E\end{pmatrix}\begin{pmatrix}\psi_+\\ \psi_-\end{pmatrix}. \tag{71}$$

Here $m(x)$ is a real white noise process with mean m_0. We have recycled the symbol m to conserve the finite resources of the alphabet. As noted in the Introduction, this model is related to a model of glass first analyzed by Dyson[14] and to a special version of the random bond Ising model introduced by McCoy and Wu.[15] It has been extensively studied previously in its various incarnations[16,13] and it is known to have a critical point at $m_0=0$ and $E=0$. Here we shall use the Landauer conductance to investigate this critical point. For simplicity, the discussion is limited to two circumstances. First we set $E=0$ and study the behavior of the conductance as $m_0\to 0$. Next we set $m_0=0$ and tune the Fermi energy, E.

A. Elementary solution for $E=0$

The techniques developed in this paper can be used to calculate the disorder averaged Landauer conductance of this model; but for the special case $E=0$ a much more complete solution can be obtained by elementary means. The content of this solution is instructive and it is therefore described.

When $E=0$ the transfer matrix of Eq. (71) has a particularly simple explicit form for arbitrary $m(x)$. Define

$$M\equiv\int_{x_i}^{x_f}dx\,m(x). \tag{72}$$

Then the transfer matrix

$$T=\begin{pmatrix}\cosh M & \sinh M\\ \sinh M & \cosh M\end{pmatrix}. \tag{73}$$

Use of Eqs. (4), (5), and (73) immediately yields

$$t=\operatorname{sech} M \tag{74}$$

and

$$g=\frac{e^2}{h}\operatorname{sech}^2 M. \tag{75}$$

Since m is a Gaussian white noise process M is a Gaussian random variable. If we suppose

$$[m]_{\rm ens}=m_0$$

and

$$[\delta m(x)\,\delta m(x')]_{\rm ens}=D\,\delta(x-x'), \tag{76}$$

where $\delta m(x)\equiv m(x)-m_0$, it is easy to calculate

$$[M]_{\rm ens}=m_0(x_f-x_i)$$

and

$$[(\delta M)^2]_{\rm ens}=D(x_f-x_i). \tag{77}$$

From Eq. (77) the full distribution of M can be reconstructed:

$$P(M)=\frac{1}{\sqrt{2\pi D(x_f-x_i)}}\exp\left(-\frac{[M-m_0(x_f-x_i)]^2}{2D(x_f-x_i)}\right). \tag{78}$$

It is now altogether straightforward to obtain statistical information about the Landauer conductance from Eqs. (75) and (78). In particular the following can be shown: (i) At the critical point the average conductance decays as a power of the system size $(x_f-x_i)^{-1/2}$. (ii) Away from the critical point $(m_0\neq 0)$ the average conductance decays exponentially (Anderson localization). The localization length diverges as $m_0^{-\nu}$ where the exponent $\nu=2$. (iii) The conductance has a very broad distribution; the average is dominated by large rare fluctuations. (iv) Away from the critical point the typical conductance also decays exponentially but with a distinct localization length that diverges with exponent $\nu=1$ as the critical point is approached.

In summary although Eq. (71) is essentially trivial to solve for $E=0$, the content of the solution is not trivial. It illustrates many general features of random critical systems.[16]

B. Tuning Fermi energy

1. Disorder averaging

The localization length is expected to diverge as $(\ln E)^2$ as $E\to 0$. To probe this singularity we shall make one simplification. The frequency E will be considered to be complex, and we shall approach the singularity at the origin along the imaginary axis. Thus we shall replace $E\to i\omega$ in Eq. (71) and study $[g]_{\rm ens}(\omega)$.

Note that for imaginary frequency Eq. (71) is real and therefore the scattering amplitude t is also real. In general t is not a suitable probe of delocalized behavior as it decays exponentially (due to phase fluctuations) even when $|t|^2$ does not. In this case, however, we may look for the transition by calculating the average of t, which is not only real, but also positive (and therefore free of sign fluctuations[23]). This constitutes a simplification because to calculate t the A fermions and bosons are not needed.

Even if one insists on calculating $|t|^2$, the A fermions and bosons are unneeded. Since t is real $|t|^2=t^2$ and we may write

$$|t|^2=\langle 0|c_+^R S_F^R(x_f,x_i)c_+^{R\dagger}|0\rangle\langle 0|b_+^R S_B^R(x_f,x_i)b_+^{R\dagger}|0\rangle. \tag{79}$$

Having noted these possibilities, we now disregard them. Instead we use Eq. (44) to calculate $|t|^2$. Hence we need to average the S matrix $S_{\rm SUSY}(x_f,x_i)$ over the white noise ensemble of real $m(x)$ with $[m(x)]_{\rm ens}=0$ and $[m(x),m(x')]_{\rm ens}=D\,\delta(x-x')$. The result is

$$[S_{\rm SUSY}(x_f,x_i)]_{\rm ens}=\exp\left(-\frac{D}{2}H_D(x_f-x_i)\right), \tag{80}$$

where

$$H_D\equiv\omega\hat{N}+\frac{D}{2}\hat{K}^2. \tag{81}$$

$\hat{N}$ counts the total number of particles regardless of species and $K\equiv i(c_+^{R\dagger}c_-^R+c_-^Rc_+^R+c_+^{A\dagger}c_-^{A\dagger}+c_-^Ac_+^A-b_+^{R\dagger}b_-^{R\dagger}+b_-^Rb_+^R-b_+^{A\dagger}b_-^{A\dagger}+b_-^Ab_+^A)$. The ensemble averaged conductance is then given by

$$[g]_{\rm ens}=\frac{e^2}{h}\langle 0|c_+^Ac_+^R\exp\left(-\frac{D}{2}H_D(x_f-x_i)\right)c_+^{R\dagger}c_+^{A\dagger}|0\rangle. \tag{82}$$

2. Critical behavior

Once again all the fermionic bilinears in $\hat{K}^2$ annihilate $c_+^{R\dagger}c_+^{A\dagger}|0\rangle$; hence the conductance is determined by the boson vacuum amplitude

$$[g]_{\rm ens}=\frac{e^2}{h}\exp[-2\omega(x_f-x_i)]\times\langle 0|_B\exp\left(-\frac{D}{2}H_D^{\rm boson}(x_f-x_i)\right)|0\rangle_B. \tag{83}$$

As before $H_D^{\rm boson}$ is Hermitean and positive definite. It has an invariant subspace spanned by the states $|n\rangle_B$, $n=0,1,2,\ldots$. $|n\rangle_B$ is here defined as the state with $2n$ bosons of each kind. Hence we may focus on the block of $H_D^{\rm boson}$ that lies within the subspace spanned by $|n\rangle_B$.

Our plan is to find eigenstates of this block and to expand the boson vacuum in terms of these eigenstates. In contrast to the two-edge Chalker-Coddington case the eigenstates are discrete and can be labeled by an integer $l=1,2,3,\ldots$. The eigenvalue problem is solved approximately in the relevant small ω limit in the next subsubsection. It will be found that the eigenvalues are given by

$$\lambda_l = \frac{2\pi^2 l^2}{(\ln\omega/D)^2} \tag{84}$$

and the overlap with the vacuum is

$$|\langle l|0\rangle_B|^2 = \frac{1}{|2\ln(\omega/D)|}\,\frac{\sinh\pi k_l}{\pi k_l}, \tag{85}$$

where $k_l \equiv l\pi/\ln(\omega/D)$. Putting together these results

$$[g]_{\rm ens} \approx \frac{e^2}{h}\,\frac{1}{|2\ln(\omega/D)|}\exp\left(-2\pi^2 D\frac{1}{[\ln(\omega/D)]^2}(x_f - x_i)\right) \tag{86}$$

for large samples. Equation (86) shows that the sample is Anderson localized away from the critical point. The localization length diverges as $(\ln\omega)^2$ as $\omega\to 0$, in agreement with previous work (Balents and Fisher[13] and references therein).

3. Solution of boson problem

This calculation is very similar to the corresponding calculation by Balents and Fisher[13] of the average Green's function . For this reason, after pointing out the special feature of the conductance calculation [discussion leading to Eq. (89)] we present the remaining steps in outline. The reader interested in more details should consult the appropriate sections of Balents and Fisher.

We wish to solve

$$H_D^{\rm boson}|\psi\rangle = \lambda|\psi\rangle \tag{87}$$

within the subspace spanned by the states $|n\rangle_B$ defined in the previous subsubsection. To this end expand $|\psi\rangle$ as

$$|\psi\rangle = a_0|0\rangle_B + a_1|1\rangle_B + \cdots + a_n|n\rangle_B + \cdots. \tag{88}$$

A difficulty arises if we assume the states are normalized so that $\langle n|_B|m\rangle_B = \delta_{mn}$. In this case the effect of $H_D^{\rm boson}$ on $|n\rangle_B$ is to yield a linear combination of $|n+1\rangle_B$, $|n\rangle_B$, and $|n-1\rangle_B$ with coefficients that involve products such as $\sqrt{n(n+1)}$. Such noninteger coefficients would then also appear when Eq. (87) is written as a recurrence relation analogous to Eq. (61) and would defeat the generating function technique that was used to solve Eq. (61).

However, if we work with an unusual normalization such that $\langle n|_B|m\rangle_B = (n+1)\,\delta_{mn}$ simple coefficients result. Adopting this convention we find

$$\left(\frac{4\omega}{D}m + 4m^2 + 4m + 1\right)a_m - m(2m-1)a_{m-1} - (m+1)(2m+3)a_{m+1} = \lambda a_m. \tag{89}$$

The corresponding generating function obeys a differential equation with three singular points, two regular and one irregular. Thus the solutions are related not to the common functions of mathematical physics but to the more obscure Mathieu or Spheroidal functions. Rather than pursue this direction we closely follow Balents and Fisher to obtain an approximate solution to the Schrödinger Eq. (89), valid in the interesting limit of small ω/D.

First we solve Eq. (89) for $\omega = 0$ using the generating function method. This solution should be accurate for $n \ll D/\omega$. Setting $a_0 = 1$ we find

$$a_n \approx \frac{1}{2n}\left(\frac{\sinh\pi k}{\pi k}\right)^{1/2}\cos[k\ln n - \phi(k)] \tag{90}$$

for $1 \ll n \ll D/\omega$. Here we have introduced k via $\lambda = 2k^2$ and $\phi(k) \equiv \arg\Gamma(1-ik) + 2k\ln 2$. Next we approximate the large-n behavior by taking the continuum limit of Eq. (89):

$$n^2\frac{d^2}{dn^2}a + 3n\frac{d}{dn}a + \left(\frac{\lambda}{2} + 1 - \frac{\omega}{D}n\right)a = 0. \tag{91}$$

Equation (91) is soluble by Laplace transforms; but an even quicker solution may be effected by introducing $f(n) \equiv na(n)$, which obeys the equation

$$n^2\frac{d^2}{dn^2}f + n\frac{d}{dn}f + \left(\frac{\lambda}{2} - \frac{\omega}{D}n\right)f = 0. \tag{92}$$

Precisely this equation was analyzed by Balents and Fisher in Appendix D of their paper. Borrowing their results we can find the values of λ for which Eq. (91) has a solution that decays as $n\to\infty$ and smoothly matches Eq. (90) for $1 \ll n \ll D/\omega$. This yields the quantized energy levels of Eq. (84).

The eigenstates may be approximately normalized by taking the continuum solution to apply everywhere. The normalization sum $\Sigma_{n=0}^{\infty} a_n^2/(n+1)$ may then be replaced by an integral. The relevant integral is evaluated asymptotically by Balents and Fisher (Appendix D). Transcription of their result yields Eq. (85).

C. Summary

For reference we summarize the results of this section.

At the critical point $(m_0 = 0, \omega = 0)$ the average conductance decays as the inverse square root of the sample size. Off the critical point the average conductance decays exponentially on a length scale called the localization length. For $\omega = 0$ we find the localization length of the average conductance diverges as $m_0^{-\nu}$ as $m_0 \to 0$, with exponent $\nu = 2$; for $m_0 = 0$, we find it diverges as $(\ln\omega)^2$. These results for the average conductance localization length are consistent with previous calculations of other non-local correlation functions for these models.[16,13]

For $\omega = 0$, the model is completely soluble by elementary methods. In this simple case it shows many of the features generally expected of random critical systems.[16] In particular the Landauer conductance is very broadly distributed and the average is dominated by large, rare fluctuations. Away from the critical point the typical conductance also decays exponentially but with a localization length different from the average conductance. The localization length for the typical conductance diverges with exponent 1 as $m_0 \to 0$.

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APPENDIX A

The purpose of this appendix is to show that the results of Sec. III are not modified when $E \neq 0$. This can be done in various ways. The method followed here is chosen because it also illustrates how to average the S matrix when the random Hamiltonian, H_{SUSY}, is composed of two noncommuting pieces.

Factorize the S matrix as

$$S_{\mathrm{SUSY}}(x,x_i)=\exp[iE\hat{M}(x-x_i)]U(x,x_i). \tag{A1}$$

Then Eqs. (42) and (43) imply that U obeys

$$-i\frac{\partial}{\partial x}U(x,x_i)=[m(x)\hat{A}(x)+m^*(x)\hat{B}]U(x,x_i) \tag{A2}$$

and the initial condition $U(x_i,x_i)=1$. Here

$$\hat{A}(x)\equiv\exp[-iE\hat{M}(x-x_i)]\hat{A}\exp[iE\hat{M}(x-x_i)],$$

$$\hat{B}(x)\equiv\exp[-iE\hat{M}(x-x_i)]\hat{B}\exp[iE\hat{M}(x-x_i)]. \tag{A3}$$

In this interaction representation it is easy to show that

$$\hat{A}(x)=\exp[-i2E(x-x_i)]\hat{A},$$

$$\hat{B}(x)=\exp[i2E(x-x_i)]\hat{B}. \tag{A4}$$

The disorder average of U can be computed as in Sec. III.

$$[U(x,x_i)]_{\mathrm{ens}}=\exp\left(-\frac{D}{2}(x-x_i)(\hat{A}\hat{B}+\hat{B}\hat{A})\right). \tag{A5}$$

Recall $\hat{A}\hat{B}+\hat{B}\hat{A}\equiv H_{CC}$ [Eq. (49)]. Hence

$$[S_{\mathrm{SUSY}}(x,x_i)]_{\mathrm{ens}}=\exp[iE\hat{M}(x-x_i)]\times\exp\left(-\frac{D}{2}H_{CC}(x-x_i)\right). \tag{A6}$$

Finally note

$$\hat{M}c_+^{R\dagger}c_+^{A\dagger}|0\rangle=0. \tag{A7}$$

Equations (44), (A6), and (A7) together show that

$$[g]_{\mathrm{ens}}=\frac{e^2}{h}\langle 0|c_+^Ac_+^R\exp\left(-\frac{D}{2}H_{CC}(x_f-x_i)\right)c_+^{R\dagger}c_+^{A\dagger}|0\rangle, \tag{A8}$$

independent of E as claimed in the paper.

APPENDIX B

The contour integral in Eq. (65) is evaluated here asymptotically for large n. $f(x)$ is defined in Eq. (64). As in the paper, we focus on $\lambda>1/2$ and write $\lambda=(1+k^2)/2$. Thus $\mu=-1/2+ik/2$.

Use the integral representation of the hypergeometric function

$$F(a,b,c;x)=\frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\int_1^\infty dt(t-x)^{-a}t^{a-c}(t-1)^{c-b-1} \tag{B1}$$

valid provided $\mathrm{Re}c>\mathrm{Re}b>0$ (Ref. 17, chapter 5).

Substitute Eq. (B1) in Eq. (65) with $a\to\mu+1$, $b\to\mu+1$ and $c\to 1$. Exchange the order of the t and x integrals to obtain

$$a_n=\frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\int_1^\infty dtt^{a-c}(t-1)^{c-b-1}f_n(t), \tag{B2}$$

where

$$f_n(t)=\oint_C\frac{dx}{2\pi i}\frac{1}{x^{n+1}}(t-x)^{-1/2-ik/2}(1-x)^{-1/2+ik/2}. \tag{B3}$$

Here C is a contour that encircles the origin but not the branch point at $x=1$.

Note that the integrand in Eq. (B3) has branch points at $x=1$ and $x=t$ where t is some point on the positive real axis to the right of 1 (and to be eventually integrated over the range from 1 to ∞). To be consistent with the conventions of the integral representation, Eq. (B1), we draw branch cuts along the positive real axis from 1 to $+\infty$ and t to $+\infty$. Also the phase of $(t-x)$ and $(1-x)$ must both be taken to be zero when x lies on the real axis to the left of 1.

The contour C is now deformed to pass above and below the branch cut. It is closed by a small circle around $x=1$ and by a big circle at ∞. The contribution of the circles to the contour integral vanishes. The contribution from integrating above and below the branch cut is

$$f_n(t)=\frac{1}{\pi}\cosh\frac{\pi k}{2}\int_1^t dx\frac{1}{x^{n+1}}(t-x)^{-1/2-ik/2}(x-1)^{-1/2+ik/2}. \tag{B4}$$

Once again exchange the order of the x and t integrals to obtain

$$a_n=\frac{1}{\pi^2}\cosh^2\frac{\pi k}{2}\int_1^\infty dx\frac{1}{x^{n+1}}(x-1)^{-1/2+ik/2}\times\int_x^\infty dtt^{-1/2+ik/2}(t-1)^{-1/2-ik/2}(t-x)^{-1/2-ik/2}. \tag{B5}$$

The Γ function prefactors in the integral representation, Eq. (B1), have been simplified by use of the formula $\Gamma(z)\Gamma(1-z)=\pi/\sin(\pi z)$.

Rescale the t integral so the lower limit becomes 1. Upon comparison with the integral representation, Eq. (B1), it is seen that

$$a_n=\frac{1}{\pi}\cosh\frac{\pi k}{2}\int_1^\infty dx\frac{1}{x^{n+1}}(x-1)^{-1/2+ik/2}x^{-1/2-ik/2}$$

$$\times F\left(\frac{1}{2}+i\frac{k}{2},\frac{1}{2}+i\frac{k}{2},1;\frac{1}{x}\right). \tag{B6}$$

Since we are interested in large-n behavior we introduce $s=\ln x$ and obtain

$$a_n=\frac{1}{\pi}\cosh\frac{\pi k}{2}\int_0^\infty ds e^{-ns}(e^s-1)^{-1/2+ik/2}(e^s)^{-1/2-ik/2}$$
$$\times F\left(\frac{1}{2}+i\frac{k}{2},\frac{1}{2}+i\frac{k}{2},1;e^{-s}\right). \tag{B7}$$

Up to this point all the manipulations have been exact. Equations (B7) shows that to obtain the large-n asymptotic behavior only the small s behavior of the various factors in the integrand is needed. The hypergeometric function has a simple expansion about zero (the hypergeometric series); however, we need to expand about one (since $s\approx 0$ implies $e^{-s}\approx 1$). This is accomplished by use of the joining formula (Ref. 17)

$$F(a,b,c;x)=\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}F(a,b,a+b-c+1;1-x)$$
$$+\frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)}(1-x)^{c-a-b}$$
$$\times F(c-a,c-b,c-a-b+1;1-x). \tag{B8}$$

We obtain

$$F\left(\frac{1}{2}+i\frac{k}{2},\frac{1}{2}+i\frac{k}{2},1;e^{-s}\right)\approx\frac{\Gamma(-ik)}{\Gamma^2(1/2-ik/2)}$$
$$+\frac{\Gamma(ik)}{\Gamma^2(1/2+ik/2)}s^{-ik}. \tag{B9}$$

Substitute Eq. (B9) in Eq. (B7). This leads to the result of Eq. (66).

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[18] It is not essential for our approach to assume $m(x)=0$ in the probes. However, it simplifies the analysis. It would be expected that the transport properties of the sample should not be modified by the nature of the probes. This can be checked explicitly under some circumstances.

[19] Perhaps this is a good place to emphasize that in this paper although the ''Hamiltonian'' is frequently non-Hermitean, it is always a well defined operator in a well defined Hilbert space. There are no difficulties of interpretation as our ''Hamiltonian'' is merely used as a calculational device. The true Schrödinger equation of the model, Eq. (1), is governed by a Hermitean Hamiltonian.

[20] The next two subsections follow Feynman very closely. For this reason we have given them the same titles as the corresponding appendices in Feynman's paper.

[21] This result is not surprising when one considers the analogous result of Strum-Liouville theory where it is known that the overlap integral of distinct eigenfunctions can be expressed as a surface term. See, for example, Ref. 17, pp. 719 and 720.

[22] Used, for example, to obtain the orthonormalization of plane waves.

[23] Proof: Consider a series of truncated problems as in Sec. II C. First note that for $x_0=x_f$, $t(x_0)=1$. Then observe that $t(x_0)$ can never vanish; if it did, the wave function vanishes in the drain, and by integrating Eq. (71) backwards, it vanishes in the sample and source also. Since it cannot pass through zero, by continuity, $t(x_0)$ must remain positive for all x_0 including x_i.