Aharonov-Bohm radiation

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A solenoid oscillating in vacuum will pair produce charged particles due to the Aharonov-Bohm (AB) interaction. We calculate the radiation pattern and power emitted for charged scalar particles. We extend the solenoid analysis to cosmic strings and find enhanced radiation from cusps and kinks on loops. We argue by analogy with the electromagnetic AB interaction that cosmic strings should emit photons due to the gravitational AB interaction of fields in the conical spacetime of a cosmic string. We calculate the emission from a kink and find that it is of similar order as emission from a cusp, but kinks are vastly more numerous than cusps and may provide a more interesting observational signature.

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I. INTRODUCTION

Fifty years ago Aharonov-Bohm (AB) [1] showed that charged particles can scatter nontrivially in the pure gauge potential outside a thin solenoid. The effect has been experimentally investigated in various setups [2,3] and quite recently (theoretically) in graphene [4]. In cosmology, AB scattering is relevant to the dynamics of cosmic strings, as they move in the ambient cosmological medium [5,6] and the AB interaction could be a link between dark strings and the visible sector [7].

AB scattering is purely quantum, having no classical analog, and also purely topological, as it arises from the nontrivial change in the phase of the wave function of a charged particle as it is taken in a closed path around the solenoid. In this paper we study if the AB interaction can also lead to particle creation—if a thin long solenoid oscillates, does it produce electron-positron pairs? Such radiation would be purely quantum and purely topological.

The question of AB radiation has been partially addressed by Alford and Wilczek [6] where the problem is set up (but not solved). In this paper, we shall take up this task using two approaches. The first is to use the AB phase, defined as $\epsilon = e\Phi$, where e is the charge and Φ the magnetic flux through the solenoid, as a small control parameter. Here the calculation can be done using the Feynman diagram language with the gauge potential of the moving solenoid treated as a classical background. The method can also be applied to find the AB radiation rate from oscillating cosmic string loops. The disadvantage of this perturbative approach is that we lose track of the expected periodicity of all quantities in the AB phase. For example, the classic "$\sin(\pi\alpha)$" behavior of AB scattering, where $\alpha = \epsilon/2\pi$, does not appear in this treatment. Our second approach is to allow the AB phase to take any value but to only consider slowly oscillating solenoids i.e. we expand in the velocity of the solenoid. The advantage now is that we explicitly see the periodic dependence of the radiation rate on the AB phase. However, we cannot treat

relativistic motion such as that of cosmic string loops in this approximation. We can, of course, take both the AB phase and the oscillation speed to be small, in which case both methods should agree, and we show that they do.

Another novel aspect of AB radiation is that it may also apply in the gravitational context. In a conical metric, such as that of a cosmic string, the wave function of any particle that goes around the string acquires a phase equal to the conical deficit, which we denote as $\delta = 8\pi G\mu$, where G is Newton's gravitational constant and μ is the string tension. Then, if a string oscillates, it emits all particles including, for example, photons. This may make cosmic strings, even those whose constituent fields have no interactions with photons, visible in the electromagnetic domain. We will discuss gravitational AB radiation in Sec. V where we rely on the analysis developed in the earlier sections. Although we arrived at gravitational AB radiation via the gauge AB process, the effect is just that of particle production in the time-dependent gravitational background of a cosmic string considered by Garriga, Harari, and Verdaguer in [8] (also see [9]).

The rest of the paper is organized so that we first discuss the gauge potential produced by a moving solenoid in Sec. II. In Sec. III we construct the Feynman diagram and scattering amplitude relevant to the rest of the calculation. Then we find the AB radiation from an infinite, straight, oscillating solenoid in Sec. IV, using first the small AB phase approximation and then in the slow velocity approximation. In Sec. V we calculate AB radiation from cosmic string loops, providing two explicit examples, one of a loop with kinks and the other of a loop with cusps but no kinks. In Sec. VI we discuss gravitational AB radiation. We conclude in Sec. VII. The formalism to find AB radiation in slowly changing backgrounds is set up in the Appendix.

In this paper we have restricted ourselves to AB radiation of spin zero (scalar) particles and are mostly concerned with the case when the emitted particles are massless. The extension to fermions is conceptually similar

but the spin orientation provides another degree of freedom and makes the calculations more involved at a technical level. We will address the fermion calculation in a separate publication [10].

II. GAUGE POTENTIAL OF MOVING SOLENOID

The first step in calculating AB radiation is to find the gauge potential for a moving solenoid. This step was taken by Alford and Wilczek in [6].

A moving solenoid with magnetic flux Φ will be a current source in Maxwell's equation, and the current can be written as

$$J_\nu^{(\Phi)} = \frac{\Phi}{2} \epsilon_{\mu\nu\alpha\beta} \partial^\mu S^{\alpha\beta}, \tag{1}$$

where

$$\begin{aligned}
S^{\alpha\beta}(x) &= \int d\tau d\sigma \sqrt{-\gamma} \epsilon^{ab} \partial_a X^\alpha \partial_b X^\beta \delta^{(4)}(x - X(\sigma, \tau)) \\
&= \int d\tau d\sigma (\dot{X}^\alpha X^{\beta\prime} - \dot{X}^\beta X^{\alpha\prime}) \delta^{(4)}(x - X(\sigma, \tau)),
\end{aligned} \tag{2}$$

where σ, τ are world-sheet coordinates, $X^\mu(\sigma)$ denotes the position of the string, overdots and primes denote derivatives with respect to τ and σ, respectively, and γ_{ab} is the world-sheet metric, with γ denoting its determinant. In what follows, we shall use $\tau = t$, while $\sigma = z$ for a solenoid along the z axis. In applications to cosmic strings, we will also use the gauge conditions

$$\dot{X} \cdot X' = 0, \qquad \dot{X}^2 + X'^2 = 0. \tag{3}$$

The field strength of a static, thin solenoid satisfies

$$\partial^\mu F_{\mu\nu} = J_\nu \tag{4}$$

and the solution is

$$F_{\mu\nu} = \frac{\Phi}{2} \epsilon_{\mu\nu\alpha\beta} S^{\alpha\beta}. \tag{5}$$

The corresponding gauge potential can be written in Lorenz gauge

$$A_\nu = \frac{\Phi}{2} \epsilon_{\mu\nu\alpha\beta} \partial^\mu \frac{1}{\partial^2} S^{\alpha\beta}, \tag{6}$$

where ∂^2 is the d'Alembertian operator. We will only need the gauge potential in momentum space,

$$\tilde{A}_\nu = -i\frac{\Phi}{2} \epsilon_{\mu\nu\alpha\beta} \frac{k^\mu}{k^2} \tilde{S}^{\alpha\beta}, \tag{7}$$

where overtildes denote the Fourier transformed variable

$$\tilde{S}^{\alpha\beta}(k) = \int d^4x e^{+ik\cdot x} S^{\alpha\beta}(x). \tag{8}$$

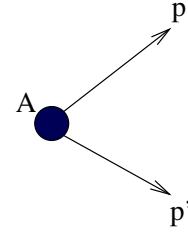Note from Eq. (5) that the field strength vanishes everywhere except at the location of the solenoid and so a

FIG. 1 (color online). Feynman diagram for pair creation by oscillating solenoid. The solenoid creates an oscillating gauge potential, A_μ, while the field strength is always zero outside the solenoid. The time-dependent pure gauge A_μ leads to particle production.

moving solenoid does not radiate classical electromagnetic waves.

III. AB RADIATION

Consider the interaction term $J_\mu A^\mu$ where, for example, for a complex scalar field, χ, we have

$$J_\mu = ie(\chi^* \partial_\mu \chi - \chi \partial_\mu \chi^*), \tag{9}$$

where e is the charge of χ. The lowest order amplitude for χ production is found from the diagram in Fig. 1 and can be written as

$$i\mathcal{M} = -i\frac{\Phi}{2} \epsilon_{\mu\nu\alpha\beta} \frac{k^\mu}{k^2} \tilde{S}^{\alpha\beta}(k) J^\nu(p, p')|_{k=p+p'} \tag{10}$$

with $J^\nu(p, p') = ie(p - p')^\nu$. Then

$$\begin{aligned}
i\mathcal{M} = &\frac{\epsilon}{2} \epsilon_{\mu\nu\alpha\beta} \frac{(p + p')^\mu (p - p')^\nu}{(p + p')^2} \\
&\times \int d\tau d\sigma (\dot{X}^\alpha X^{\beta\prime} - \dot{X}^\beta X^{\alpha\prime}) e^{-i(p+p')\cdot X}, \tag{11}
\end{aligned}$$

where $\epsilon = e\Phi$ is the AB phase and is assumed to be small.

The number of particles produced is given by [see, for example, Eq. (4.74) of [11]]

$$dN = \frac{d^3p}{(2\pi)^3} \frac{1}{2\omega} \frac{d^3p'}{(2\pi)^3} \frac{1}{2\omega'} |\mathcal{M}|^2, \tag{12}$$

where $\omega = \sqrt{p^2 + m^2}$ (similarly ω') and m is the mass of the scalar particle.

IV. AB RADIATION FROM OSCILLATING SOLENOID

In this section we will consider AB radiation from an infinite straight solenoid that is oscillating back and forth. We will do the calculation in the small AB phase approximation. The oscillating solenoid is also treated in the nonrelativistic approximation in Sec. IV B, and the results of the two methods agree in the regime where both are valid.

We will take the position of the solenoid to be described by

$$X^\mu = (t, A\sin(\Omega t), 0, \sigma), \tag{13}$$

where A is the amplitude and Ω the frequency of oscillation. Note that X^μ is not the solution to any equation of motion but just a convenient oscillatory choice that provides us with a background in which to study AB radiation.

A. Small AB phase approximation

Inserting Eq. (13) in Eq. (11) gives

$$i\mathcal{M} = -\frac{2\epsilon}{(p+p')^2}[(p_x p'_y - p_y p'_x)I_0 + (\omega' p_y - \omega p'_y)I_1], \tag{14}$$

where $\omega = p^0$, $\omega' = p^{0\prime}$,

$$I^0 = \int d\tau d\sigma e^{-i(p+p')\cdot X}, \tag{15}$$

$$I^1 = \int d\tau d\sigma \Omega A \cos(\Omega\tau) e^{-i(p+p')\cdot X}. \tag{16}$$

The integration along the straight string is trivial and leads to a $2\pi\delta(p_z + p'_z)$ factor in I^0 and I^1.

To simplify further, we use the relation

$$e^{i\alpha\sin(\Omega\tau)} = \sum_{n=-\infty}^{+\infty} J_n(\alpha)e^{in\Omega\tau}, \tag{17}$$

which can be checked using the integral representation for the Bessel functions

$$J_n(\alpha) = \frac{1}{2\pi}\int_{-\pi}^{+\pi} e^{-i(n\tau - \alpha\sin\tau)}d\tau. \tag{18}$$

Then

$$I^0 = \frac{(2\pi)^2}{\Omega}\sum_n J_n(P^1 A)\delta(n - (P^0/\Omega))\delta(p_z + p'_z), \tag{19}$$

and

$$I^1 = 2\pi^2 A\sum_n [J_{n-1}(P^1 A) + J_{n+1}(P^1 A)]$$
$$\times \delta(n - (P^0/\Omega))\delta(p_z + p'_z)$$
$$= \frac{(2\pi)^2}{P^1}\sum_n n J_n(P^1 A)\delta(n - (P^0/\Omega))\delta(p_z + p'_z), \tag{20}$$

where $P = p + p'$ and $J_n(x)$ is the Bessel function of nth order.

We can now insert the expressions for the integrals I^0 and I^1 in Eq. (14) to get

$$i\mathcal{M} = -8\pi^2\epsilon\frac{(p_y - p'_y)}{(p_x + p'_x)}\sum_{n=1}^{\infty} J_n((p_x + p'_x)A)$$
$$\times \delta(\omega + \omega' - n\Omega)\delta(p_z + p'_z). \tag{21}$$

At first glance the amplitude depends on six variables, namely, the six different momentum components.

However, Eq. (21) shows that it only depends on the combinations $p_x + p'_x$, $p_y - p'_y$, and $p_z + p'_z$ as long as the overall energy is a harmonic of the oscillation frequency. Note that the amplitude is not singular at $p_x + p'_x = 0$ because the Bessel functions are proportional to $(p_x + p'_x)^n$ at small argument and the sum in Eq. (21) starts at $n = 1$ [see Fig. 2 for plots of $J_n(x)/x$]. The amplitude is largest when $p_y - p'_y$ is maximum and $p_x + p'_x = 0 = p_z + p'_z$. Together with the total energy constraint

$$\omega + \omega' = \sqrt{p^2 + m^2} + \sqrt{p'^2 + m^2} = n\Omega, \tag{22}$$

we find that the maximum amplitude is along the y direction, with $p_x = 0 = p'_x$.

The number of particles emitted follows from Eq. (12). On squaring $\mathcal{M}$ we get a double sum but the energy conservation delta function reduces the expression to a single sum and introduces a $\delta(0)$ on the right-hand side. Dividing out by $2\pi\delta(0)$ gives the *rate* of particle production, denoted by $\dot{N}$,

$$d\dot{N} = \frac{d^3 p}{(2\pi)^3}\frac{1}{2\omega}\frac{d^3 p'}{(2\pi)^3}\frac{1}{2\omega'}\frac{(8\pi^2\epsilon)^2}{2\pi}\left(\frac{p_y - p'_y}{p_x + p'_x}\right)^2$$
$$\times \sum_{n=1}^{\infty} |J_n((p_x + p'_x)A)|^2$$
$$\times \delta(\omega + \omega' - n\Omega)\{\delta(p_z + p'_z)\}^2. \tag{23}$$

The integration over p'_z leads to another factor of $\delta(0)$. Dividing out by another factor of $2\pi\delta(0)$ gives the rate of particle emission *per unit length* of the solenoid, denoted by $\dot{N}'$,

$$d\dot{N}' = \frac{\epsilon^2}{16\pi^4}dp_z d^2 p_\perp d^2 p'_\perp \frac{1}{\omega\omega'}\left(\frac{p_y - p'_y}{p_x + p'_x}\right)^2$$
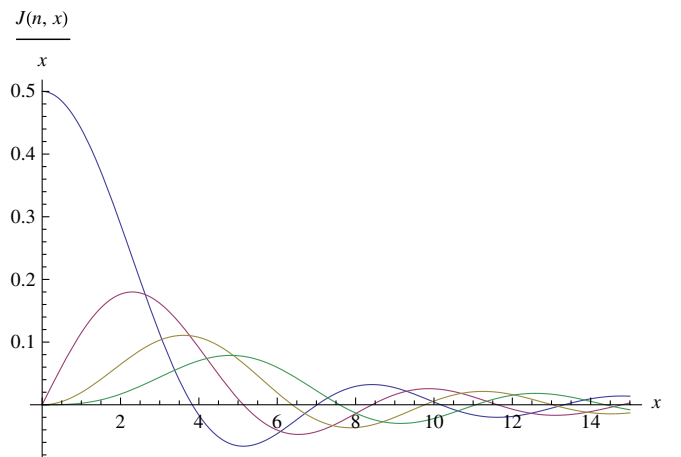$$\times \sum_{n=1}^{\infty} |J_n((p_x + p'_x)A)|^2\delta(\omega + \omega' - n\Omega), \tag{24}$$

FIG. 2 (color online). Plot of $J_n(x)/x$ for $n = 1, 2, 3,$ and 4, where the amplitude of the peak decreases with increasing n.

where the $\perp$ subscript denotes the x, y components. Also, now $\omega' = \sqrt{p_\perp'^2 + p_z^2 + m^2}$.

The expression for the rate of particle emission depends on five of the momenta. The dependence on p_z enters via ω and ω', and it is clear that the largest emission is for $p_z = 0$ i.e. in the plane perpendicular to the solenoid. So we restrict our attention to $p_z = 0$ and also to the massless case, $m = 0$. These assumptions imply that $\omega = |\boldsymbol{p}_\perp|$, $\omega' = |\boldsymbol{p}_\perp'|$, and we obtain in polar coordinates

$$\frac{d\dot{N}'}{dp_z}(p_z = 0) = \frac{\epsilon^2}{16\pi^4} dp_\perp dp_\perp' d\theta d\theta' \left(\frac{p_y - p_y'}{p_x + p_x'}\right)^2$$

$$\times \sum_{n=1}^{\infty} |J_n((p_x + p_x')A)|^2$$

$$\times \delta(p_\perp + p_\perp' - n\Omega), \qquad (25)$$

where $p_x = p_\perp \cos\theta$ etc.

The integration over $p_\perp'$ can be done giving the emission rate per unit length in the orthogonal plane as a function of the three variables, $p_\perp$, θ, and θ'. The angular distribution is of greatest interest and is shown for $n = 1$ and $n = 2$ in Fig. 3, taking $p_\perp = n\Omega/2$. The figure shows back to back emission, with the $n = 1$ emission being dipolar and maximum along the y direction.

The total radiation rate can be found by integrating Eq. (24) over momenta and also doing the sum over harmonics. The sum is convergent because the Bessel functions at fixed argument fall off exponentially with n for large n. We will work only in the $m = 0$ case and restrict attention to particle emission in the orthogonal plane, as in Eq. (25). This is equivalent to working out the problem in two spatial dimensions, and we shall denote the particle emission rate by $N^{(2d)}$. The delta function can be integrated out by going to polar coordinates. We get

$$\dot{N}^{(2d)} = \frac{\epsilon^2 \Omega}{16\pi^4} \sum_{n=1}^{\infty} nC_n(a_n), \qquad (26)$$

where $a_n \equiv n\Omega A$ and

$$C_n(a) = \int_0^1 dq \int_{-\pi}^{\pi} d\theta \int_{-\pi}^{\pi} d\theta' \left(\frac{q\sin\theta - (1-q)\sin\theta'}{q\cos\theta + (1-q)\cos\theta'}\right)^2$$

$$\times |J_n(a(q\cos\theta + (1-q)\cos\theta'))|^2. \qquad (27)$$

These integrations have to be done numerically.

The energy radiation rate in the nth harmonic is

$$\dot{E}_n = n\Omega\dot{N}_n, \qquad (28)$$

and Fig. 4 shows a plot of $\ln(\dot{\mathcal{E}}_n)$ versus n, where $\dot{\mathcal{E}}_n = \dot{E}_n 16\pi^4/(\epsilon^2\Omega^2)$ for $\Omega A = 0.1$, 0.5, and 1. Note that the energy falls off rapidly with increasing n and so essentially all of the particles are emitted in the $n = 1$ harmonic i.e. with energy Ω.

The total energy emission rate $\dot{\mathcal{E}}$ as a function of the parameter ΩA is shown in Fig. 5.
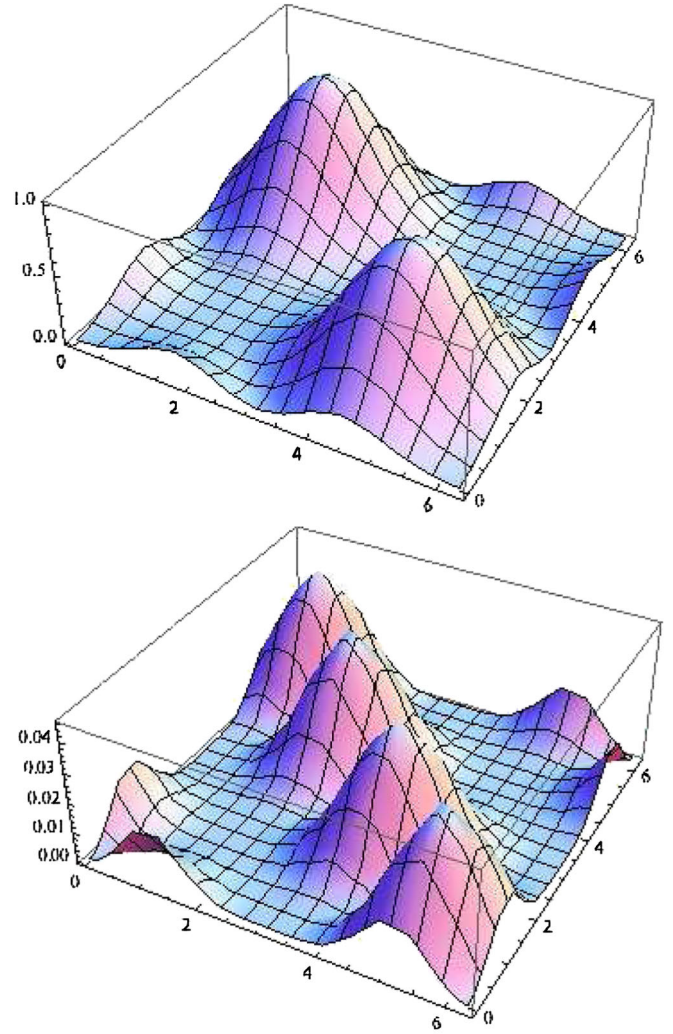

FIG. 3 (color online). Here we show the radiated power as a function of θ and θ'. The top figure is for $n = 1$ with $|\boldsymbol{p}| = n\Omega/2 = |\boldsymbol{p}'|$ and the lower figure is for $n = 2$. There is maximum radiation along the line $\theta + \theta' = \pi$, implying back to back emission. For $n = 1$, the peaks occur at $\theta = \pi/2$, $3\pi/2$. For $n = 2$, they occur at $\theta = \pi/4$, $3\pi/4$, $5\pi/4$, and $7\pi/4$.
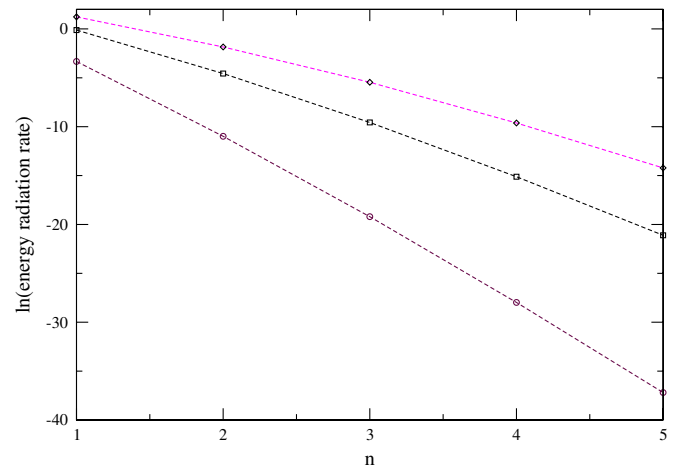

FIG. 4 (color online). $\ln(\dot{\mathcal{E}}_n)$ vs n for $\Omega A = 0.1$ (lowest curve), 0.5 (middle curve), and 1.0 (top curve).
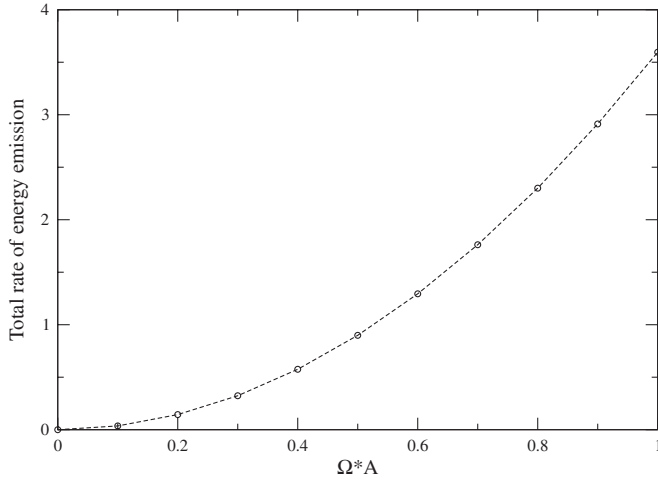
FIG. 5. $\dot{\mathcal{E}}$ vs ΩA for the oscillating solenoid.

We have not calculated the emission rate for massive particles, $m \neq 0$, in detail. From Eq. (23) we see that m enters through ω and ω' in the phase space factor and then again in the energy conservation delta function but nowhere else. The delta function can be nonzero only if $n\Omega > 2m$ i.e. $n > 2m/\Omega$. If the oscillation frequency is small compared to the mass of the particle—as is relevant for oscillating solenoids in the laboratory when the charged particle is the electron—the Bessel functions decay exponentially with large n and we expect the emission to be exponentially suppressed. The exponential suppression is also seen in the plot of Fig. 4.

It is noteworthy that the total energy of the pairs produced add up to $n\Omega$ by virtue of the delta function constraint in Eq. (24). Commonly in quantum mechanics, perturbations at frequency Ω produce excitations with energy $\hbar\Omega$ (as, for example, in optical transitions in atomic physics). Here excitations with energies equal to all multiples of the fundamental frequency are produced (although in practice the higher harmonic production may be small except in an ultrarelativistic limit). This situation is also encountered in laboratory scale phenomena such as Mossbauer spectroscopy and the excitation of conduction electrons by the motion of impurities in metals. Mathematically the appearance of excitations at higher harmonics is traceable to the Fourier expansion Eq. (17).

In the next section, using the formalism developed in the Appendix, we reanalyze AB radiation from an oscillating solenoid for arbitrary values of the AB phase, ϵ, but in the limit of small ΩA. We recover the periodicity of the result as a function of the AB phase. We also show that the results of the two different methods agree in the limit that both ϵ and ΩA are small.

B. Small oscillation frequency approximation

Now let us consider the circumstance that the flux is not a small perturbation. In this case we can still obtain ana-

lytic results for particle production in the limit that the solenoid motion is nonrelativistic. These results are nonperturbative in the flux and show the flux periodicity expected of the Aharonov-Bohm effect. Moreover, the small flux limit of the results obtained in this section match the low velocity limit of the results in the previous section.

The essential physics here is that for a stationary solenoid the eigenmodes of the scalar field depend (periodically) on the flux Φ of the solenoid. If the solenoid is moved to a different location the eigenmodes and hence the quantum vacuum is altered. If the solenoid is moved slowly from the initial to the final configuration, in the adiabatic approximation it will simply pass from the initial vacuum to the final vacuum, but postadiabatic corrections lead to particle production. The technical implementation of this idea is relegated to the Appendix where we analyze the leading postadiabatic dynamics of an assembly of oscillators with slowly varying eigenmodes.

We will work in cylindrical coordinates and in the "singular gauge" where the gauge potential of a static solenoid is

$$A^\mu = (0, 0, \Phi\Theta(x)\delta(y), 0). \tag{29}$$

Here Θ is the Heaviside function. The gauge potential vanishes everywhere on the xy plane and is singular along the positive x axis. It is easy to check that the line integral of the gauge potential along a path enclosing the origin yields the flux Φ.

The eigenmodes, ψ_{kql}, for a scalar field in the background of a solenoid along the z axis have been found in Ref. [1]. We will use those eigenmodes in singular gauge. In order to box normalize the eigenmodes, we adopt periodic boundary conditions along the z axis and take the length of the solenoid to be L. Along the radial direction we find it convenient to impose Dirichlet boundary conditions at $r = a$ [i.e. $\psi_{kql}(a) = 0$] if l is even, and Neumann boundary conditions [i.e. $\partial_r \psi_{kql}(a) = 0$] if l is odd. These boundary conditions are convenient because they allow us to do certain integrals in closed form; they also give the physical result that the particle production vanishes in the limit that the solenoid flux goes to zero. The values of k now take on discrete values for finite a, but the set becomes a continuum in the limit $a \to \infty$. Then the singular gauge eigenmodes are

$$\psi_{kql}(r, \theta, z) = \sqrt{\frac{k}{2La}} J_{|l-\alpha|}(kr)e^{iqz}e^{i(l-\alpha)\theta}, \tag{30}$$

where $0 \leq \theta < 2\pi$, and $\alpha = \epsilon/2\pi$. The modes are labeled by the radial momentum k, the z momentum, q, and the azimuthal quantum number l. The frequency of the mode is

$$\omega_{kq} = \sqrt{k^2 + q^2 + m^2}. \tag{31}$$

Note that the eigenmodes have a discontinuity along the positive x axis. This is because we are working in singular

gauge where the gauge potential is singular on the positive x axis.

The normalization integral of the modes is

$$\int_0^a dr r \int_0^{2\pi} d\theta \int_0^L dz \psi^*_{k'q'l'} \psi_{kql} = \delta_{ll'} \delta_{qq'} \delta_{kk'}, \quad (32)$$

where we have taken $ka \gg |(l - \alpha)^2 - 1/4|$ and used the asymptotic form of the Bessel functions

$$J_\nu(x) \approx \sqrt{\frac{2}{\pi x}} \cos\left(x - \frac{\nu \pi}{2} - \frac{\pi}{4}\right) \quad (33)$$

together with the integral

$$\int_0^a dr r J_\nu(kr) J_\nu(k'r) = \frac{a^2}{2}[(J_\nu(ka))^2 - J_{\nu-1}(ka)J_{\nu+1}(ka)] \\ \times \delta_{kk'}. \quad (34)$$

As in the previous section we assume that the solenoid oscillates in the xz plane while remaining aligned with the z axis. The modes for a shifted solenoid are obtained by appropriately translating the modes Eq. (30). Hence the transition element designated $a_{\beta\alpha}$ in the Appendix, Eqs. (A6) and (A15), is here given by

$$a_{k'q'l',kql} = \int_0^a dr r \int_0^{2\pi} d\theta \int_0^L dz \psi^*_{k'q'l'} \frac{\partial}{\partial x} \psi_{kql}. \quad (35)$$

A straightforward calculation reveals

$$a_{k'q'l',kql} = -L\delta_{qq'} \frac{2k \sin(\pi\alpha)}{k'^2 - k^2} \left(\frac{k'}{k}\right)^\alpha \quad (36)$$

for $l = 0$, $l' = 1$, and

$$a_{k'q'l',kql} = -L\delta_{qq'} \frac{2k \sin(\pi\alpha)}{k'^2 - k^2} \left(\frac{k'}{k}\right)^\alpha \quad (37)$$

for $l = 1$, $l' = 0$; for all other combinations of l and l' it vanishes.

Using Eq. (A16) and taking the continuum limit $L, a \to \infty$ we obtain $d\dot{N}' = dk dq \, I(k, q)$, where

$$I(k, q) = \frac{v_0^2 \sin^2(\pi\alpha)}{16\pi^2} \frac{k k_c^2}{\omega_{kq} \Omega^2} \left[\left(\frac{k}{k_c}\right)^{2\alpha} + \left(\frac{k_c}{k}\right)^{2\alpha-2}\right] \\ \times \Theta(\Omega - \omega_{kq} - \sqrt{q^2 + m^2}) \quad (38)$$

is the rate of particle production at wave vector (k, q). (To include antiparticles, we would double this rate.) Here $k_c \equiv (\Omega^2 + k^2 - 2\Omega\omega_{kq})^{1/2}$. Equation (38) applies for $0 < \alpha < 1$; by virtue of the periodicity of the scalar field modes, $I(k, q)$ is a periodic function of α with period 1.

Some of the quantities in Eq. (38) have a straightforward interpretation in terms of the kinematics of pair production. Let the pair of particles have momenta (k, q) and (k', q'). By conservation of z-axis momentum $q + q' = 0$. By energy conservation $\omega_{kq} = \Omega - \omega_{k'q'}$. The minimum value of $\omega_{k'q'}$ is $\sqrt{q^2 + m^2}$; hence $\omega_{kq} < \Omega - \sqrt{q^2 + m^2}$; this

accounts for the Θ function in Eq. (38). Similarly if one particle has momentum (k, q), then k_c represents the radial momentum of the second particle.

In the limit $\alpha \to 0$ Eq. (38) simplifies to

$$I(k, q) = \frac{\alpha^2}{16} \frac{v_0^2}{\Omega^2} \frac{k}{\omega_{kq}} (k_c^2 + k^2) \\ \times \Theta(\Omega - \omega_{kq} - \sqrt{m^2 + q^2}). \quad (39)$$

To make contact with the results of the previous section we consider the nonrelativistic limit of the result for $d\dot{N}'$ [Eq. (24)] in which only the first harmonic contribution is significant. If we write $\boldsymbol{p}$ in cylindrical coordinates as $\boldsymbol{p} = (k, \theta, q)$, integrate $d\dot{N}'$ over $\boldsymbol{p}'$ and θ, halve the result since the relativistic expression counts production of particles and antiparticles, we obtain the flux in Eq. (39). Thus the nonrelativistic limit of the small flux approximation coincides with the small flux limit of the nonrelativistic moving solenoid approximation, providing a useful check on both calculations.

V. AB RADIATION FROM COSMIC STRING LOOPS

A cosmic string loop trajectory is written in terms of left and right movers

$$X = \tfrac{1}{2}[\boldsymbol{a}(\sigma - \tau) + \boldsymbol{b}(\sigma + \tau)]. \quad (40)$$

The choice of gauge, Eq. (3), requires

$$|\boldsymbol{a}'| = 1 = |\boldsymbol{b}'|, \quad (41)$$

where primes denote derivatives with respect to the argument.

Insertion of Eq. (40) in (11) with some simplifications gives

$$i\mathcal{M} = \frac{\epsilon}{4} \epsilon_{\mu\nu\alpha\beta} \frac{(p + p')^\mu (p - p')^\nu}{(p + p')^2} I_+^\alpha I_-^\beta, \quad (42)$$

where

$$I_+^\alpha = \int_{-\infty}^{+\infty} d\sigma_+ b^{\alpha\prime} e^{-ik\cdot b/2}, \\ I_-^\alpha = \int_{-\infty}^{+\infty} d\sigma_- a^{\alpha\prime} e^{-ik\cdot a/2}, \quad (43)$$

and $\sigma_\pm \equiv \sigma \pm \tau$ and $k = p + p'$. Note that the integrations have been extended to the entire σ, t plane. This is valid since the integrands are periodic functions in both σ and t, and the only effect of the larger domain of integration will be to yield Dirac delta functions instead of Kronecker deltas.

The integrals $I_\pm$ satisfy the orthogonality relations

$$k \cdot I_\pm = 0, \quad (44)$$

which allow us to relate $I_\pm^0$ to $\boldsymbol{I}_\pm$,

$$k^0 I_\pm^0 = \mathbf{k} \cdot \mathbf{I}_\pm. \qquad (45)$$

By using these relations, we find

$$S \equiv \epsilon_{\mu\nu\alpha\beta} p^{\mu\prime} p^\nu I_+^\alpha I_-^\beta = \frac{k_\mu k^\mu}{2k^0}(\mathbf{p} - \mathbf{p}') \cdot (\mathbf{I}_+ \times \mathbf{I}_-). \qquad (46)$$

Now we can write

$$|\mathcal{M}|^2 = \frac{\epsilon^2}{4(k_\mu k^\mu)^2}|S|^2$$

$$= \frac{\epsilon^2}{16(\omega + \omega')^2}|(\mathbf{p} - \mathbf{p}') \cdot \mathbf{I}_+ \times \mathbf{I}_-|^2. \qquad (47)$$

The integrations in Eqs. (43) extend over an infinite range whereas the loop dynamics is periodic. We can use the series representation of the Dirac delta function

$$\delta(x) = \frac{1}{2\pi} \sum_{n=-\infty}^{+\infty} e^{inx}, \qquad -\pi < x < +\pi \qquad (48)$$

to write the integrals over only one periodic domain

$$I_+^\alpha = \sum_{n=1}^{\infty} \int_0^L d\sigma_+ b^{\alpha\prime} e^{-ik\cdot b/2} \delta\left(\frac{k^0 L}{4\pi} - n\right),$$

$$I_-^\alpha = \sum_{n=1}^{\infty} \int_0^L d\sigma_- a^{\alpha\prime} e^{-ik\cdot a/2} \delta\left(\frac{k^0 L}{4\pi} - n\right). \qquad (49)$$

The integrals $\mathbf{I}_\pm$ are nonvanishing only for discrete positive values of the total energy

$$k^0 = \omega + \omega' = \frac{4\pi n}{L}, \qquad (50)$$

where n is a positive integer. Therefore it is convenient to write the squared amplitude for a fixed harmonic

$$|\mathcal{M}|_n^2 = \left(\frac{\epsilon L}{16\pi n}\right)^2 |(\mathbf{p} - \mathbf{p}') \cdot \mathbf{I}_+^{(n)} \times \mathbf{I}_-^{(n)}|^2, \qquad (51)$$

where $\mathbf{I}_\pm^{(n)}$ refer to the integrals in Eq. (49) together with the constraint in Eq. (50).

The squared amplitude can now be written as

$$|\mathcal{M}|^2 = \frac{\pi}{L} \sum_{n=1}^{\infty} |\mathcal{M}|_n^2 \delta\left(\frac{k^0 L}{4\pi} - n\right)\delta_E(0), \qquad (52)$$

where $\delta_E(0)$ is an energy conservation delta function and we will divide out by $2\pi\delta_E(0)$ to obtain particle radiation per unit time.

We will now find the radiation in the $m = 0$ case from two different kinds of loops, one that contains kinks (sharp corners where the tangent vector is discontinuous) and the other that contains cusps (points on the loop that momentarily reach the speed of light).

A. Kinky loop

Here we consider the "degenerate kinky loop"

$$\begin{aligned} \mathbf{a} &= \sigma_- \mathbf{A}, & 0 \le \sigma_- \le L/2, \\ \mathbf{a} &= (L - \sigma_-)\mathbf{A}, & L/2 \le \sigma_- \le L, \\ \mathbf{b} &= \sigma_+ \mathbf{B}, & 0 \le \sigma_+ \le L/2, \\ \mathbf{b} &= (L - \sigma_+)\mathbf{B}, & L/2 \le \sigma_+ \le L, \end{aligned} \qquad (53)$$

where $\mathbf{A}$ and $\mathbf{B}$ are two fixed unit vectors. Also note that

$$a^0 = -\sigma_-, \qquad b^0 = \sigma_+, \qquad (54)$$

so that $X^0 = t$.

This loop is "degenerate" because it consists of four straight segments and is "kinky" because of its four corners. The four straight segments propagate with constant speed but shrink and expand due to the motion of the kinks (see Fig. 6).

Now the relevant integrals can be evaluated in the $m = 0$ case and with $k^0 = 4\pi n/L$,

$$\mathbf{I}_-^{(n)} = \frac{2L}{n\pi} \frac{e^{i(1+\boldsymbol{\kappa}\cdot\mathbf{A})n\pi/2}}{1 - (\boldsymbol{\kappa}\cdot\mathbf{A})^2} \sin\left((1 + \boldsymbol{\kappa}\cdot\mathbf{A})\frac{n\pi}{2}\right)\mathbf{A}, \qquad (55)$$

$$\mathbf{I}_+^{(n)} = \frac{2L}{n\pi} \frac{e^{-i(1-\boldsymbol{\kappa}\cdot\mathbf{B})n\pi/2}}{1 - (\boldsymbol{\kappa}\cdot\mathbf{B})^2} \sin\left((1 - \boldsymbol{\kappa}\cdot\mathbf{B})\frac{n\pi}{2}\right)\mathbf{B}, \qquad (56)$$

where $\boldsymbol{\kappa} \equiv \mathbf{k}/k^0 = L(\mathbf{p} + \mathbf{p}')/4\pi n$. This gives

$$|\mathcal{M}|_{m=0;n}^2 = \frac{\epsilon^2}{16}\left(\frac{L}{n\pi}\right)^6 |(\mathbf{p} - \mathbf{p}') \cdot \mathbf{A} \times \mathbf{B}|^2$$
$$\times \frac{\sin^2((1 + \boldsymbol{\kappa}\cdot\mathbf{A})\frac{n\pi}{2})\sin^2((1 - \boldsymbol{\kappa}\cdot\mathbf{B})\frac{n\pi}{2})}{[1 - (\boldsymbol{\kappa}\cdot\mathbf{A})^2]^2[1 - (\boldsymbol{\kappa}\cdot\mathbf{B})^2]^2}. \qquad (57)$$

The expression in (57) is finite as can be seen by taking limits, say as $\boldsymbol{\kappa} \cdot \mathbf{A} \to 1$. This shows that kinks on loops give off finite AB radiation at every harmonic. (The sum over harmonics, however, will be seen to diverge.) Maximum emission occurs when $\mathbf{p} - \mathbf{p}'$ is parallel to $\mathbf{A} \times \mathbf{B}$ i.e. perpendicular to the plane of the loop. Also, the factors depending on $\boldsymbol{\kappa}$ are maximum for $\boldsymbol{\kappa} \cdot \mathbf{A} = 0 = \boldsymbol{\kappa} \cdot \mathbf{B}$, implying that $\boldsymbol{\kappa}$, hence $\mathbf{p} + \mathbf{p}'$, is perpendicular to the plane of the loop. Since the amplitude vanishes for $\mathbf{p} - \mathbf{p}' = 0$, we conclude that the maximum emission occurs for $\mathbf{p} = -\mathbf{p}'$ and perpendicular to the plane of the loop.
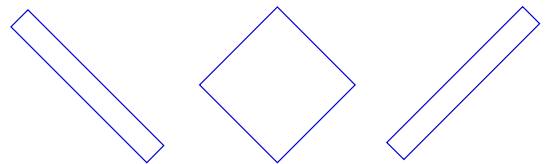

FIG. 6 (color online). Three snapshots of a degenerate kinky loop.

The differential rate of particle production is given by Eq. (12)

$$d\dot{N} = \frac{1}{2L}\frac{d^3p}{(2\pi)^3}\frac{1}{2\omega}\frac{d^3p'}{(2\pi)^3}\frac{1}{2\omega'}$$

$$\times \sum_{n=1}^{\infty}|\mathcal{M}|_{m=0;n}^2\delta(n-(\omega+\omega')L/4\pi). \quad (58)$$

The total energy emitted is

$$\dot{E} = \sum_{n=1}^{\infty}\dot{E}_n, \quad (59)$$

where

$$\dot{E}_n = \frac{2\pi}{L^2}n\int\frac{d^3p}{(2\pi)^3}\frac{1}{2\omega}\int\frac{d^3p'}{(2\pi)^3}\frac{1}{2\omega'}$$

$$\times |\mathcal{M}|_{m=0;n}^2\delta(n-(\omega+\omega')L/4\pi). \quad (60)$$

The integral can be evaluated by numerical means.

However it is prudent to check the scaling with n before performing detailed calculation. Because of the energy conservation delta function we estimate $\omega \sim n$, $\omega' \sim n$, $d^3p \sim n^3$, and $d^3p' \sim n^3$, and the delta function contributes n^{-1}. The contribution of the factor $|\mathcal{M}|_{m=0;n}^2$ is less clear: a factor of n^{-4} is left over after rescaling all the momenta and there may also be factors of n due to contributions from the trignometric functions as suggested by considering the similar integral

$$I = \int_0^1 dx\frac{\sin^2(nx)}{x^2}$$

$$= n\int_0^n dy\frac{\sin^2(y)}{y^2} \rightarrow n\int_0^{\infty} dy\frac{\sin^2(y)}{y^2} = O(n^{+1}). \quad (61)$$

To find the scaling, we have summed $\dot{E}_n$ over n up to a cutoff N using

$$\sum_{n=1}^{N}\sin^2(nx)\sin^2(ny) = \frac{M}{8} - \frac{\sin(Mx)}{8\sin x} - \frac{\sin(My)}{8\sin y}$$

$$+ \frac{\sin(Mx_-)}{16\sin x_-} + \frac{\sin(Mx_+)}{16\sin x_+}, \quad (62)$$

where $M = 2N + 1$ and $x_{\pm} = x \pm y$. Then the remaining integrations are done numerically on MATHEMATICA to evaluate

$$P(N) = \sum_{n=1}^{N}\dot{E}_n.$$

In Fig. 7, we plot $P(N)$ versus N for the loop with $\mathbf{A} = \hat{x}$ and $\mathbf{B} = \hat{y}$, clearly showing that the total power radiated grows linearly with the upper cutoff N.

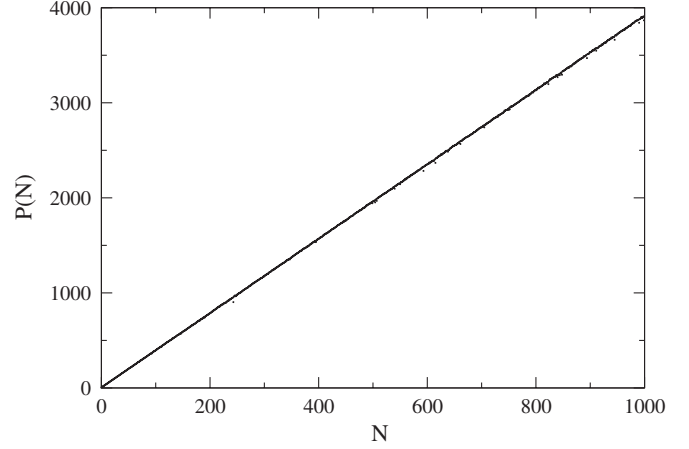Since $P(N) \propto N$, the total power emitted diverges linearly and

FIG. 7. The plot of $2\pi(\pi^2L/4\epsilon)^2P(N)$ versus N for the degenerate kinky loop shows $P(N) \propto N$ growth.

$$P(N) \sim \frac{\epsilon^2}{L^2}N. \quad (63)$$

The kink width provides a cutoff on harmonics and we assume this is similar to the width of the string, $1/\sqrt{\mu}$. Therefore it is reasonable to take $N \sim \sqrt{\mu}L$ and estimate the total power emitted,

$$\dot{E} \sim \epsilon^2\mu\left(\frac{1}{\sqrt{\mu}L}\right). \quad (64)$$

This energy emission rate is much larger than the naive dimensional analysis result ϵ^2/L^2, by a factor of $\sqrt{\mu}L \sim L/w$, where w is the width of the string.

B. Cuspy loop

We now consider smooth loops that have cusps i.e. points that reach the speed of light.

A simple loop trajectory is given by

$$a^0 = -\sigma_-, \qquad \mathbf{a} = \frac{L}{2\pi}(\sin s_-, -\cos s_-, 0),$$

$$b^0 = \sigma_+, \qquad \mathbf{b} = \frac{L}{2\pi}(\sin s_+, 0, -\cos s_+), \quad (65)$$

where $s_{\pm} = 2\pi\sigma_{\pm}/L$. The loop has cusps because there are points such that $\mathbf{a}' = -\mathbf{b}'$. These occur at $s_+ = s_- = 0$, π, at which point the velocity of the string is $\mathbf{v} = \pm\hat{x}$ i.e. the point on the string reaches the speed of light.

Inserting the loop trajectory in Eq. (49) gives

$$\underline{I}_-^{(n)} = (-1)^{n+1}e^{in\phi_-}L\left[\frac{J_n(n\kappa_{xy})}{\kappa_{xy}}\hat{\boldsymbol{\kappa}}_{xy} - iJ_n'(n\kappa_{xy})\hat{\boldsymbol{\kappa}}_{xy}^{\perp}\right],$$

$$\underline{I}_+^{(n)} = e^{-in\phi_+}L\left[\frac{J_n(n\kappa_{xz})}{\kappa_{xz}}\hat{\boldsymbol{\kappa}}_{xz} - iJ_n'(n\kappa_{xz})\hat{\boldsymbol{\kappa}}_{xz}^{\perp}\right], \quad (66)$$

where $\kappa^{\mu} = Lk^{\mu}/(4\pi n)$,

$$\boldsymbol{\kappa}_{xz} \equiv (\kappa_x, 0, \kappa_z), \qquad \boldsymbol{\kappa}_{xy} \equiv (\kappa_x, \kappa_y, 0),$$
$$\boldsymbol{\kappa}_{xz}^{\perp} \equiv (-\kappa_z, 0, \kappa_x), \qquad \boldsymbol{\kappa}_{xy}^{\perp} \equiv (\kappa_y, -\kappa_x, 0), \tag{67}$$

and we have defined the unit vectors $\hat{\kappa}_{xz}$, $\hat{\kappa}_{xy}$, and

$$\tan\phi_+ = \frac{\kappa_z}{\kappa_x}, \qquad \tan\phi_- = \frac{\kappa_y}{\kappa_x}. \tag{68}$$

The rate of energy emission is obtained from Eq. (60). We rescale all momenta and energy by $L/4\pi n$ and obtain

$$\dot{E}_n = \frac{\epsilon^2 n^4}{2\pi L^2} \int_0^1 dq \int d^2\hat{p} \int d^2\hat{p}' q(1-q) \left| (q\hat{p} \right.$$
$$- (1-q)\hat{p}') \cdot \left\{ \frac{J_n(n\kappa_{xy})}{\kappa_{xy}} \hat{\kappa}_{xy} - iJ_n'(n\kappa_{xy})\hat{\kappa}_{xy}^{\perp} \right\}$$
$$\left. \times \left\{ \frac{J_n(n\kappa_{xz})}{\kappa_{xz}} \hat{\kappa}_{xz} - iJ_n'(n\kappa_{xz})\hat{\kappa}_{xz}^{\perp} \right\} \right|^2, \tag{69}$$

where $\boldsymbol{\kappa} = q\hat{p} + (1-q)\hat{p}'$.

Note that the integrand is nonsingular because of the properties of the Bessel functions (see Fig. 2). The integration is over a compact region and can be evaluated numerically. For the evaluation it is advantageous to rewrite Eq. (69) using the Bessel function identities

$$2n \frac{J_n(x)}{x} = J_{n-1}(x) + J_{n+1}(x), \tag{70}$$

$$2J_n'(x) = J_{n-1}(x) - J_{n+1}(x). \tag{71}$$

To estimate the dependence of $\dot{E}_n$ on n, we compare the present integration to that for the kinky loop. The main difference is that the integrand for the kinky loop contains trignometric functions whose argument contains n, whereas here we have Bessel functions. As larger values of n are considered, the Bessel functions contribute for smaller windows of κ_{xy} and κ_{xz}, whereas the periodic trignometric functions contribute for smaller windows but there are more such windows because of the periodic nature of the trignometric functions. Following the argument below Eq. (60), we expect $\dot{E}_n \propto n^0$ and hence $P(N) \propto N$ for the cuspy loops.

We have numerically evaluated the integration in Eq. (69) and, in Fig. 8, we plot $8\pi L^2 \dot{E}_n/\epsilon^2$ versus $n = 1, \ldots, 10$. The plot confirms the $\dot{E}_n \propto n^0$ estimate for the cuspy loops.

The string thickness $w \sim 1/\sqrt{\mu}$ provides a natural cutoff on the wavelength of the emitted radiation. Thus the maximum harmonic $\sim L/w$ and we estimate the total AB power for massless particles from the cuspy loop

$$\dot{E} \sim \frac{\epsilon^2}{L^2} N \sim \frac{\epsilon^2}{L^2} \frac{L}{w}. \tag{72}$$

This is of the same order of magnitude as the radiation from kinks.
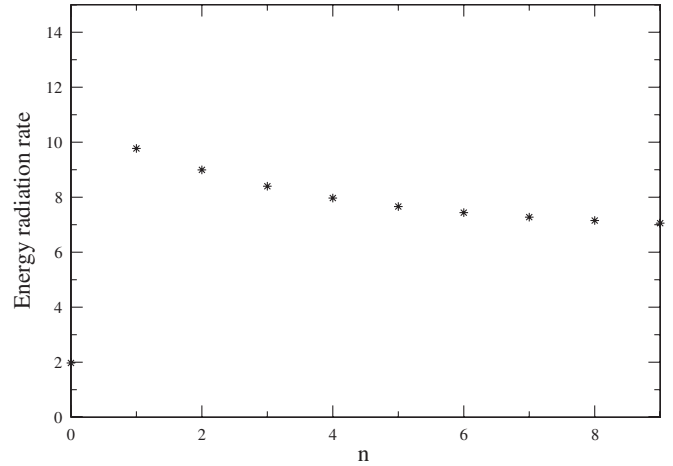

FIG. 8. $(8\pi L^2/\epsilon^2)\dot{E}_n$ versus n for the loop with cusps.

The emission of massive particles is conceptually similar but technically more involved. We have left that analysis for future work.

VI. GRAVITATIONAL AB RADIATION

The above analysis for AB radiation suggests that a corresponding effect must exist in a gravitational setting. Consider the conical metric of a straight cosmic string (see Fig. 9). When *any* particle goes around the string and returns to its original position, the wave function acquires a phase factor given by the conical deficit angle $\delta = 8\pi G\mu$, where μ is the string tension. Hence any particle will interact with a cosmic string by the gravitational AB effect and we expect an oscillating cosmic string to radiate gravitational AB radiation that contains *all* species of particles.

The existence of gravitational AB radiation may also be argued on the grounds that an oscillating cosmic string provides a time-dependent background for quantum fields. All the vacuum modes of the quantum fields have to adjust to the dynamical background. Then it is natural to expect some "sloshing" i.e. pair creation. The process has been studied in [8] and here we extend that analysis to the case of kinks.

The most intriguing aspect of gravitational AB radiation is that it implies that cosmic strings will radiate photons.
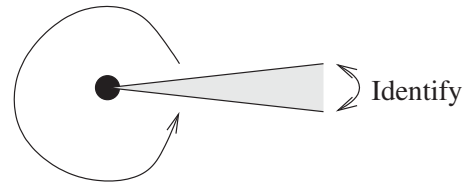

FIG. 9. A conical space is obtained by cutting out a wedge from a Euclidean two-dimensional space and identifying the edges. The wave function of a particle that goes around the tip of the cone acquires a phase given by the deficit angle.

Based on our calculations of the previous sections, we expect the emission to be largest from kinks on loops. We will now sketch a calculation of the gravitational AB radiation of massless scalars from a degenerate, kinky loop. The relevant interaction between the metric and the scalar field can be obtained if we start with the action for a massless scalar field

$$S_\phi = \int d^4x \sqrt{-g}\, \frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi \qquad (73)$$

and expand around the Minkowski metric,

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \qquad g^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu}. \qquad (74)$$

This gives

$$S_\phi = S_{\phi,0} - \int d^4x \frac{1}{2} H^{\mu\nu} \partial_\mu \phi \partial_\nu \phi + \cdots, \qquad (75)$$

where $S_{\phi,0}$ is the action in a Minkowski background, and

$$H_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu} h_\alpha^\alpha. \qquad (76)$$

The metric perturbation, $h_{\mu\nu}$, is sourced by the string and can be written in Fourier space as

$$\tilde{h}_{\mu\nu} = -\frac{16\pi G}{k^2}\left(\tilde{T}_{\mu\nu}^{(s)} - \frac{1}{2}\eta_{\mu\nu}\tilde{T}_\alpha^{(s)\alpha}\right), \qquad (77)$$

where the superscript (s) on $\tilde{T}$ denotes that this is the Fourier transform of the string energy-momentum tensor which is given by [12]

$$T_{\mu\nu}^{(s)}(x) = \mu \int d^2\sigma(\dot{X}_\mu \dot{X}_\nu - X'_\mu X'_\nu)\delta^4(x - X). \qquad (78)$$

Now the string energy-momentum tensor can be written in terms of $a(\sigma_-)$ and $b(\sigma_+)$.

$$T^{(s)\mu\nu} = -\frac{\mu}{4}\int d^2\sigma_\pm(a^{\mu\prime}b^{\nu\prime} + a^{\nu\prime}b^{\mu\prime})\delta^4(x - X). \qquad (79)$$

The Fourier transform can be done and it is clear that it will factorize into integrations over left-moving (σ_-) and right-moving (σ_+) variables. For our present analysis, we would like to focus on the degenerate, kinky loops, for which $a^{\mu\prime}$ and $b^{\mu\prime}$ are piecewise continuous. Then

$$\tilde{T}^{(sq)\mu\nu} = -\frac{\mu}{4}(I_+^\mu I_-^\nu + I_+^\nu I_-^\mu), \qquad (80)$$

where $I_\pm^\mu$ have the same meaning as in Eq. (43).

In momentum space we have

$$\tilde{H}_{\mu\nu} = -\frac{16\pi G}{k^2}\tilde{T}_{\mu\nu}, \qquad (81)$$

and the pair production amplitude from the square loop is

$$i\mathcal{M}^{(sq)} = -\frac{8\pi G}{k^2}\tilde{T}_{\mu\nu}^{(sq)}(p^\mu p^{\nu\prime} + p^\nu p^{\mu\prime})$$

$$= \frac{\delta}{8 p \cdot p'}(p \cdot I_+ p' \cdot I_- + p \cdot I_- p' \cdot I_+). \qquad (82)$$

We use the orthogonality relations in Eqs. (44), together with energy conservation, to simplify the amplitude expression as in Sec. VA, to eventually obtain

$$|\mathcal{M}^{(sq)}|^2 = 4\delta^2\left(\frac{L}{n\pi}\right)^4 q^2(1-q)^2[\hat{q} \cdot A\hat{q} \cdot B]^2$$

$$\times \frac{\sin^2((1 + \boldsymbol{\kappa} \cdot A)\frac{n\pi}{2})\sin^2((1 - \boldsymbol{\kappa} \cdot B)\frac{n\pi}{2})}{[1 - (\boldsymbol{\kappa} \cdot A)^2]^2[1 - (\boldsymbol{\kappa} \cdot B)^2]^2}, \qquad (83)$$

where, as in Sec. VA,

$$\boldsymbol{\kappa} = \frac{L(\boldsymbol{p} + \boldsymbol{p}')}{4\pi n}, \qquad (84)$$

and

$$q = \frac{L|\boldsymbol{p}|}{4\pi n}, \qquad \hat{q} = \frac{\hat{p}' - \hat{p}}{|\hat{p}' - \hat{p}|}. \qquad (85)$$

The expression in Eq. (83) for gravitational AB radiation from square loops is very similar to the expression for gauge AB radiation from degenerate, kinky loops, given in Eq. (57). The main difference is in the terms preceding the trignometric functions. The $[\hat{q} \cdot A\hat{q} \cdot B]^2$ factor implies that gravitational AB radiation is in the plane of the loop, in contrast to gauge AB radiation. However, the scaling with n follows the discussion below Eq. (60), and is expected to be independent of the form of the prefactor. As a check on the argument, we have evaluated $P(N)$ numerically and the results are shown in Fig. 10, confirming that $P(N) \propto N$ i.e. $\dot{E}_n \propto n^0$.

So far we have outlined gravitational AB radiation into massless scalars. We can proceed similarly for the radiation of photons. Now the linear order interaction is

$$S = \frac{1}{2}\int d^4x\left(h^{\mu\nu} - \frac{1}{4}\eta^{\mu\nu}h_\alpha^\alpha\right)F_{\mu\beta}F_\nu^\beta, \qquad (86)$$
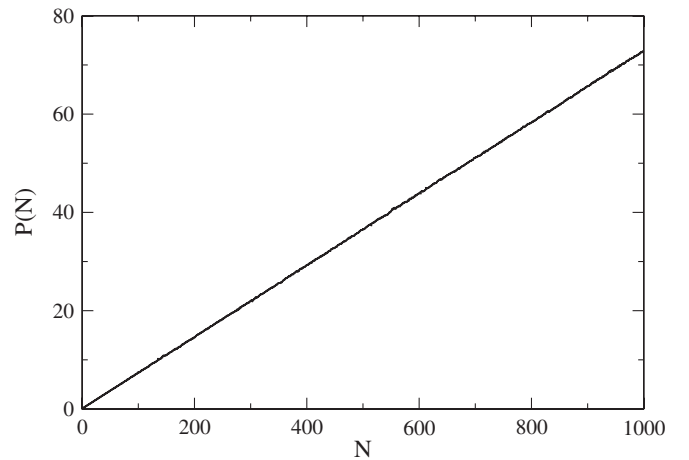

FIG. 10.   Plot of the power radiated $2\pi(\pi^2 L/8\delta)^2 P(N)$ up to some harmonic N for gravitational AB radiation from the square loop, showing the $P(N) \propto N$ growth.

where the field strength is defined in terms of the gauge potential as $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ and indices are raised and lowered by the Minkowski metric. The rest of the calculation is expected to be very similar to that for massless scalars, though the final state will also depend on photon polarizations. We expect that there will be string-like signatures contained in the polarizations, $\boldsymbol{\epsilon}$ of the emitted photons, e.g. linear polarization along the string due to factors like $\boldsymbol{\epsilon} \cdot \boldsymbol{A}$ in the amplitude, and these would be worth exploring further.

As in the gauge AB radiation case, the $P(N) \propto N$ (i.e. $\dot{E}_n \propto n^0$) result suggests that the power radiated in *photons* due to the gravitational AB effect can be estimated as

$$\dot{E}_{AB\gamma} \sim \delta^2 \mu \left(\frac{1}{\sqrt{\mu} L} \right). \tag{87}$$

The loop also emits power in gravitational waves,

$$\dot{E}_{gw} \sim \delta \mu. \tag{88}$$

Hence

$$\dot{E}_{AB\gamma} \sim \dot{E}_{gw} \delta \left(\frac{w}{L} \right), \tag{89}$$

where w denotes the width of the string. Even though the power emitted in photons is a very small fraction of the power emitted in gravitational waves, it can still be of interest, especially because it is in photons. For $G\mu = 10^{-8}$, we can write $\mu \sim 10^{51}$ erg/s, from which we get $\dot{E}_g \sim 10^{43}$ erg/s and $\dot{E}_\gamma \sim 10^{35}(w/L)$ erg/s. The lifetime of the loop is determined by its gravitational radiation rate and is estimated as $\sim L/\delta$. Hence the total energy emitted in photons is $\sim \delta \sqrt{\mu}$. With $\delta \sim 10^{-8}$, the total energy emitted in photons $\sim 10^7$ GeV and is independent of the size of the loop.

The number of photons emitted per unit time in the nth harmonic, $\dot{N}_n$, is proportional to n^{-1}. Hence the same number of photons are emitted in every logarithmic frequency interval

$$d\dot{N}_\omega \sim \frac{\delta^2}{L} \frac{d\omega}{\omega}. \tag{90}$$

In [8] the accumulated photon background due to *cusps* on a network of cosmic strings was considered. Here we estimate photon emission from kinks along a single long string in the present universe. This is relevant for direct detection of a string. The signal may be stronger than that from cusps particularly because kinks are expected to be much more numerous on strings as compared to cusps. A recent estimate of the number density of kinks on a string is [13]

$$n_k \sim \frac{1}{t} \left(\frac{t}{t_*} \right)^{0.9}, \tag{91}$$

where t_* is the epoch at which friction on the string network dynamics ceases to be important compared to the

Hubble damping. Based on the scattering rate of particles off the string, a rough estimate is $t_* \sim t_f / \delta$ where t_f is the time of formation of the strings.

To obtain numerical estimates, we have the present cosmological epoch at $t_0 \sim 10^{17}$ s, and for strings with $\delta = 10^{-8}$, we find $t_* \sim 10^{-27}$ s. Thus $t_0/t_* \sim 10^{44}$. With each kink emitting a total energy of $\delta \sqrt{\mu}$ over a Hubble time, the emission per unit length of string per unit time is $\sim \delta \sqrt{\mu}(t_0/t_*)/t_0^2 \sim 10^4$ erg/cms, which is $\sim 10^{25}$ erg/kpcs. The energy emitted from strings on astrophysical length scales is quite small compared to astrophysical sources (e.g. the solar luminosity is $\sim 10^{33}$ erg/s), though the unique stringy features (such as linear emission, polarization, flat spectrum) of the emission may help observations. Even if the signal is too weak to be used to find strings on the sky, it might be a useful signature to confirm the presence of a string if one is suspected in some field of view.

Before closing this section it is important to point out that our estimate treats all kinks equally whereas Copeland and Kibble [13] emphasize that most kinks are nearly straight and it is relevant to consider the sharpness of the kink. Hence a more rigorous estimate should fold in the kink sharpness distribution with the photon emission rate and also include effects of kink smoothing. (The analysis in [13] accounts for string straightening due to Hubble expansion but not kink smoothing due to radiation backreaction.) We leave more realistic calculations of observational signatures for future work.

VII. CONCLUSIONS

We have studied the problem of an oscillating solenoid in vacuum and determined the rate of scalar particle creation due to the Aharonov-Bohm interaction. This is novel because the electromagnetic fields of a moving solenoid vanish everywhere except at the location of the solenoid. In particular, a moving solenoid does not emit electromagnetic radiation but does emit charged particle pairs as AB radiation. We have also determined the pattern of AB radiation and find that it is predominantly in the direction perpendicular to the plane of oscillation.

Cosmic string loops that have AB interactions with quantum fields will also emit AB radiation. In the thin string limit, loops with cusps and kinks emit a linearly divergent power. Imposing a cutoff due to the thickness of the string indicates that the AB radiation is significantly stronger than the naive dimensional estimate.

We point out that all quantum fields interact with cosmic strings via the gravitational AB effect, and thus there is corresponding gravitational AB radiation of all particles, including photons. Cosmic strings, even if occurring in dark matter sectors of particle physics, will emit photons. We have calculated the energy emitted in photons from loops with kinks and cusps. As in the gauge case, the radiated energy is much larger than the naive dimensional

estimate. We have made a rough estimate of the energy emitted in photons and it remains to be seen if the signal is strong enough to make it an interesting observational tool.

APPENDIX: MOVING FRAME PERTURBATION THEORY

Consider a system of coupled oscillators governed by the time-dependent Lagrangian

$$L = \frac{1}{2}\sum_i \left(\frac{dx_i}{dt}\right)^2 - \frac{1}{2}\sum_{ij} K_{ij}(t)x_i x_j. \tag{A1}$$

The time dependence enters through the coupling matrix $K(t)$. It is helpful to determine the instantaneous normal modes $q_\alpha(t)$ with instantaneous frequencies $\omega_\alpha(t)$ that satisfy

$$\sum_j K_{ij}(t)q_{\alpha j}(t) = \omega_\alpha^2(t)q_{\alpha i}(t). \tag{A2}$$

These modes are normalized according to

$$\sum_i q_{\alpha i}q_{\beta i} = \delta_{\alpha\beta}. \tag{A3}$$

The oscillator coordinates may then be expanded

$$x_i = \sum_\alpha \xi_\alpha(t)q_{\alpha i}(t), \tag{A4}$$

where $\xi_\alpha(t)$ are instantaneous "normal mode" coordinates. The Lagrangian equation (A1) may be written in normal mode coordinates as

$$L = \frac{1}{2}\sum_\beta \left(\frac{d\xi_\beta}{dt} + A_{\beta\alpha}\xi_\alpha\right)\left(\frac{d\xi_\beta}{dt} + A_{\beta\gamma}\xi_\gamma\right) - \frac{1}{2}\sum_\alpha \omega_\alpha^2 \xi_\alpha^2, \tag{A5}$$

where a summation over the repeated indices α and γ is left implicit. Here the "transition element"

$$A_{\beta\alpha}(t) = \sum_i q_{\beta i}\frac{dq_{\alpha i}}{dt} \tag{A6}$$

is determined by the evolution of the instantaneous normal modes. Equation (A5) generalizes the textbook treatment of normal modes for coupled oscillators to the case that the Lagrangian is time dependent. In the limit that the coupling

matrix K is time independent, $A = 0$, and Eq. (A5) reduces to the textbook normal mode Lagrangian.

From the normal mode Lagrangian equation (A5) we may pass to the Hamiltonian

$$H = \frac{1}{2}\sum_\alpha [\Pi_\alpha^2 + \omega_\alpha^2 \xi_\alpha^2] - \sum_{\alpha\beta} \Pi_\alpha A_{\alpha\beta}\xi_\beta, \tag{A7}$$

where the canonical momentum is

$$\Pi_\alpha = \frac{d\xi_\alpha}{dt} + A_{\alpha\beta}\xi_\beta. \tag{A8}$$

Hamilton's equations then read

$$\begin{aligned}
\frac{d\xi_\alpha}{dt} &= \Pi_\alpha - \sum_\beta A_{\alpha\beta}\xi_\beta, \\
\frac{d\Pi_\alpha}{dt} &= \sum_\beta \Pi_\beta A_{\beta\alpha} - \omega_\alpha^2 \xi_\alpha.
\end{aligned} \tag{A9}$$

In the limit that the coupling matrix K is time independent, $A = 0$, and the normal mode coordinates simply undergo the expected harmonic oscillations. In the limit that the coupling matrix varies "slowly," the normal modes also vary slowly, allowing us to solve Hamilton's equations by treating A as a perturbation.

To zeroth order the solution is

$$\begin{aligned}
\xi_\alpha^{(0)}(t) &= \xi_\alpha(-T)\cos[\omega_\alpha(t+T)] \\
&\quad + \frac{\Pi_\alpha(-T)}{\omega_\alpha}\sin[\omega_\alpha(t+T)], \\
\Pi_\alpha^{(0)}(t) &= \Pi_\alpha(-T)\cos[\omega_\alpha(t+T)] \\
&\quad - \omega_\alpha \xi_\alpha(-T)\sin[\omega_\alpha(t+T)].
\end{aligned} \tag{A10}$$

The first-order correction is

$$\begin{aligned}
\xi_\alpha^{(1)}(t) &= \int_{-T}^t d\tau\, \xi_\beta^{(0)}(\tau)A_{\beta\alpha}(\tau)\cos[\omega_\alpha(t-\tau)] \\
&\quad + \int_{-T}^t d\tau\, \frac{1}{\omega_\alpha}\Pi_\beta^{(0)}(\tau)A_{\beta\alpha}(\tau)\sin[\omega_\alpha(t-\tau)], \\
\Pi_\alpha^{(1)}(t) &= \int_{-T}^t d\tau\, \Pi_\beta^{(0)}(\tau)A_{\beta\alpha}(\tau)\cos[\omega_\alpha(t-\tau)] \\
&\quad - \int_{-T}^t d\tau\, \omega_\alpha \xi_\beta^{(0)}(\tau)A_{\beta\alpha}(\tau)\sin[\omega_\alpha(t-\tau)].
\end{aligned} \tag{A11}$$

For simplicity in the perturbative solution Eqs. (A10) and (A11) and hereafter it is assumed the eigenfrequencies $\omega_\alpha(t)$ are independent of time although the eigenmodes $q_\alpha(t)$ may be time dependent. It is not difficult to obtain the solution in the case of time-dependent eigenfrequencies but the formulas are more cumbersome and for the application to solenoids the special formulas given here suffice. This is because as the solenoid moves the eigenmodes shift but their frequencies do not change.

Thus far we have focused on the solution to the classical equations of motion. However for a linear system such as Eq. (A11) the same solution applies in quantum mechanics if we interpret $\Pi_\alpha(t)$ and $\xi_\alpha(t)$ as operators in the Heisenberg picture that are solutions to the Heisenberg equations of motion.

To further analyze the quantum mechanics we introduce the ladder operators

$$a_\alpha(t) = \sqrt{\frac{\omega_\alpha}{2}}\xi_\alpha(t) + \frac{i}{\sqrt{2\omega_\alpha}}\Pi_\alpha(t),$$

$$a_\alpha^\dagger(t) = \sqrt{\frac{\omega_\alpha}{2}}\xi_\alpha(t) - \frac{i}{\sqrt{2\omega_\alpha}}\Pi_\alpha(t). \tag{A12}$$

Equations (A10)–(A12) show that the creation and annihilation operators at time t are related to those at the earlier time $-T$ via the canonical transformation

$$a_\alpha(t) = \sum_\beta u_{\beta\alpha}a_\beta(-T) + \sum_\beta v_{\beta\alpha}a_\beta^\dagger(-T),$$

$$a_\alpha^\dagger(t) = \sum_\beta u_{\beta\alpha}^* a_\beta^\dagger(-T) + \sum_\beta v_{\beta\alpha}^* a_\beta(-T), \tag{A13}$$

where the Bogolyubov coefficients

$$u_{\beta\alpha} = \delta_{\beta\alpha}e^{-i\omega_\alpha(t+T)} + \frac{1}{2}\left(\sqrt{\frac{\omega_\alpha}{\omega_\beta}} + \sqrt{\frac{\omega_\beta}{\omega_\alpha}}\right)e^{-i\omega_\alpha t - i\omega_\beta T}$$

$$\times \int_{-T}^t d\tau A_{\beta\alpha}(\tau)e^{-i(\omega_\beta - \omega_\alpha)\tau},$$

$$v_{\beta\alpha} = \frac{1}{2}\left(\sqrt{\frac{\omega_\alpha}{\omega_\beta}} - \sqrt{\frac{\omega_\beta}{\omega_\alpha}}\right)e^{-i\omega_\alpha t + i\omega_\beta T}$$

$$\times \int_{-T}^t d\tau A_{\beta\alpha}(\tau)e^{-i(\omega_\beta + \omega_\alpha)\tau}. \tag{A14}$$

Equations (A13) and (A14) contain complete information about quantum particle production via the dynamical Aharonov-Bohm effect. For example the number of particles in mode α at time T, $\langle n_\alpha(T)\rangle = \langle a_\alpha^\dagger(T)a_\alpha(T)\rangle$, is given by

$$\sum_\beta \frac{1}{4}\left(\sqrt{\frac{\omega_\alpha}{\omega_\beta}} - \sqrt{\frac{\omega_\beta}{\omega_\alpha}}\right)^2 \left|\int_{-T}^T d\tau A_{\beta\alpha}(\tau)e^{-i(\omega_\alpha + \omega_\beta)\tau}\right|^2.$$

If we take the transition matrix element to oscillate in time with frequency Ω,

$$A_{\beta\alpha}(\tau) = a_{\beta\alpha}\cos\Omega\tau, \tag{A15}$$

then the rate of particle production in mode α

$$R_\alpha = \frac{\pi}{8}\sum_\beta\left(\sqrt{\frac{\omega_\alpha}{\omega_\beta}} - \sqrt{\frac{\omega_\beta}{\omega_\alpha}}\right)^2|a_{\beta\alpha}|^2\delta[\Omega - (\omega_\alpha + \omega_\beta)]. \tag{A16}$$

Here we have made use of the usual golden rule prescription $T = \pi\delta[\Omega - (\omega_\alpha + \omega_\beta)]$.

Now let us adapt this general formalism to the case of a moving solenoid. In place of Eq. (A1) we have the Lagrangian

$$\mathcal{L} = \left|\frac{\partial}{\partial t}\phi\right|^2 - |(\nabla - ie\mathbf{A})\phi|^2 - m^2|\phi|^2 \tag{A17}$$

that describes a massive scalar field coupled to the vector potential of the solenoid, which is given by Eq. (29) in the singular gauge we use here. In place of Eq. (A2) the instantaneous eigenmodes satisfy

$$-[\nabla - ie\mathbf{A}(\mathbf{r}, t)]^2\Psi_\alpha(\mathbf{r}, t) + m^2\Psi_\alpha(\mathbf{r}, t) = \omega_\alpha^2\Psi_\alpha(\mathbf{r}, t). \tag{A18}$$

If the solenoid is located along the z axis the eigenmodes are given by Eq. (30). Here we consider a solenoid moving in the x-z plane while remaining parallel to the z axis; at time t its x displacement is $(v_0/\Omega)\sin\Omega t$. The eigenmodes at time t, $\Psi_\alpha(\mathbf{r}, t)$ are therefore obtained by simply displacing the eigenfunctions Eq. (30) along the x axis by the appropriate amount. Thus the modes are labeled by $\alpha \to k$, q, l and have eigenfrequency

$$\omega_{k,q,l}(t) = \sqrt{k^2 + q^2 + m^2}. \tag{A19}$$

Note that for this problem the instantaneous mode frequencies do not vary in time although the mode functions do vary in time.

Now the transition elements Eq. (A6) are given by

$$A_{k,q,l;k',q',l'}(t) = \int d\mathbf{r}\Psi_{k,q,l}^*(\mathbf{r}, t)\frac{d}{dt}\Psi_{k',q',l'}(\mathbf{r}, t)$$

$$= v_0\cos\Omega t\int d\mathbf{r}\psi_{k,q,l}^*(\mathbf{r})\frac{\partial}{\partial x}\psi_{k',q',l'}(\mathbf{r}),$$

leading to Eq. (35) for the reduced transition element $a_{k,q,l;k',q',l'}$ given in the main body of the paper. The explicit form of $a_{k,q,l;k',q',l'}$ worked out there [Eqs. (36) and (37)] reveals that particles are produced only in modes with $l = 0$ or $l = 1$.

Thus far we have focused on classical analysis. Since this is a complex field to analyze the quantum field theory we must introduce two sets of creation and annihilation operators per mode: $c_{k,q,l}^\dagger$ and $c_{k,q,l}$ that create and annihilate particles in mode k, q, l and $d_{k,q,l}^\dagger$ and $d_{k,q,l}$ that create and annihilate antiparticles.

It follows from Eq. (A16) that the rate of particle production in mode (k, q, l) is given by

$$R_{kql} = \frac{\pi}{8}\sum_{k',q',l'}\left(\sqrt{\frac{\omega_{kql}}{\omega_{k'q'l'}}} - \sqrt{\frac{\omega_{k'q'l'}}{\omega_{kql}}}\right)^2|a_{kql;k'q'l'}|^2$$

$$\times \delta[\Omega - (\omega_{kql} + \omega_{k'q'l'})]. \tag{A20}$$

Recall that we had discretized our modes for convenience by placing the scalar field in a box of dimensions a and L. Taking the continuum limit $a \to \infty$ and $L \to \infty$ in Eq. (38),

summing over the azimuthal quantum number l yields the expression for $I(k, q)$ in Eq. (38), the rate for production of a single particle species in modes with wave vectors k, q. In taking the continuum limit we make use of the usual rules

$$\frac{1}{L}\sum_q \to \int_{-\infty}^{\infty}\frac{dq}{2\pi}, \qquad \frac{1}{a}\sum_k \to \int_0^{\infty}\frac{dk}{\pi}. \tag{A21}$$

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