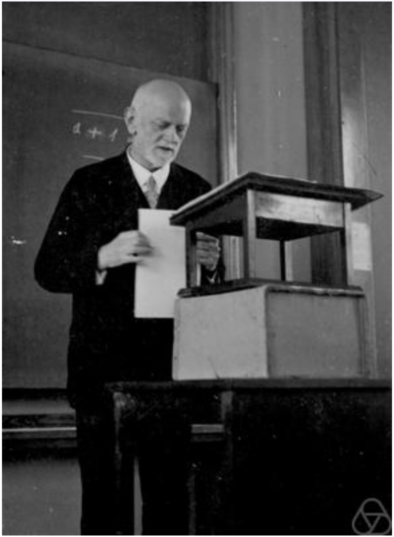

David Hilbert 1932

The question of what it takes to give proofs at all, becomes the question of what it takes to prove X.

Hilbert's insight in logic (1890s and on)

Foundations of Geometry

Foundations of Arithmetic

"One could confide Hilbert's axioms to a reasoning machine, such as the logical piano of Jevons, and one would see all of geometry come out." (Poincaré 1902).

Hilbert could not have expressed his goal that clearly in 1902.

But Poincaré spoke well.

This was Hilbert's goal, and it developed over time.

I will not linger on Formalism, which is not Hilbert's term.

Frege, Brouwer.

Hilbert 1922:
"No one can drive us from the paradise Cantor has made for us."

Hilbert 1900:

"In investigating the foundations of a science, we must set up a system of axioms. . . . But above all I wish to designate the following as the most important among the numerous questions which can be asked with regard to the axioms: To prove that they are not contradictory."

Hilbert sees better than anyone that consistency in formal systems is a question of finite strings of symbols.

Such questions are easy!

Well, they can be solved.

In terms not exactly Hilbert's:

Assumptions weaker than Peano Arithmetic will suffice to prove set theory consistent – because it is just a matter of finite symbol strings.

A great mistake.

A truly great mistake.

Gödel 1931 proves *NO* plausibly comprehensive axioms for mathematics suffice for *THEIR OWN* consistency.

Today we focus this on Peano Arithmetic.

But precisely Gödel's argument shows there is no decisive locus for consistency proofs – and so none for proof in general.

Could arithmetic possibly be inconsistent?

The question only arises for *formalized* arithmetic/set theory/....

Axiom scheme of induction: for any formula $\phi(x)$ in the language

$$[\phi(0) \wedge \forall x(\phi(x) \rightarrow \phi(sx))] \rightarrow \forall x(\phi(x))$$

A bold claim to say you grasp the expressive power of general formulas $\phi(x)$ in the language of arithmetic.

We do claim it, in some sense,....

I point I owe to John Mayberry: we have only ever applied induction for conditions ϕ that *make sense*.

If the formalism conceals an inconsistency we should expect it to lie in conditions ϕ with no apparent sense.

Set theory (any reasonable variant) *does* prove PA consistent.

As does PA plus the axiom "PA is consistent."

An empirical claim from Kreisel: We cannot prove consistency of PA *from any assumptions more plausible in the first place than consistency of* PA.

Ditto for any foundation.

Two responses to Gödel incompleteness:

Give up on foundations.

Keep track of what foundations work for what theorem.

I do not mean every mathematician should at each moment ask what axioms they are using.

Does anyone worry about the foundations of this building?

Yes, of course, many people did, and do.

Therefore you and I need not.

Read Sophus Lie, Felix Klein, Paul Gordan in the 19th century.

Read 1920s algebraic geometry, or differential geometry.

These are beautiful, and *hard*.

See how far set theory, Bourbaki, category theory, et al. have clarified math.

Prima facie two kinds of incompleteness:

The Gödel phenomenon:

> Inevitable in comprehensive foundations.
>
> Generally remote from practical questions.

Concrete incompleteness:

> E.g. ZF and the axiom of choice, PA and Goodstein or Paris-Harrington's Ramsey.
>
> Inevitable? In some way?

In other words, is this a true dichotomy?

Incompleteness is not entirely inevitable.

There are important complete theories.

Tame theories. O-minimal theories.

Many genuine problems, as in algebraic number theory or geometry, can be put in contexts free of Gödel phenomenon.

Contexts which will not (themselves) support the fearsome complexity of mathematical induction.

Precisely. Do not *forget* the phenomenon. Take care to *avoid* it.

Crucially, Gödel's theorem is *not* particularly about PA.

Robinbson's Q, EFA, Z_2, ETCS, ZFC.

Between Gödel phenomenon and concrete incompleteness:

Gödel's incompleteness theorem for PA: PA can state but not prove that there exists a function ϕ such that every theorem stated in fewer than n letters has at least one proof using fewer that $\phi(n)$ letters (indeed no such function is recursive).

Paris-Harrington incompleteness for PA: PA can state but not prove that there exists a function N such that the smallest set which solves a certain coloring problem with data n, k, m has size $N(n, k, m)$. This function is recursive but not provably so in PA.

Ramsey theorem incompleteness for EFA: the fragment EFA of PA (described below) PA can state but not prove that there exists a function N solving a somewhat simpler coloring problem. Such a function can be primitive recursive (it can be tetration= superexponential) but cannot be iterated exponential.

EFA can state but not prove solubility of the basic finite Ramsey problem: For any n, k, m find N with the following property: if we color each of the n-element subsets of $S = \{1, 2, 3, \ldots, N\}$ with k colors, then there is $Y \subseteq S$ with at least m elements, such that all n element subsets of Y have the same color.

Even PA cannot prove solubility of the Ramsey problem with the additional requirement that the smallest element of Y is at most the number of elements of Y.

The tetration function

$$^{y}2 \;=\; 2^{\wedge}(2^{\wedge}(2^{\wedge}\ldots(2^2)\ldots)) \quad \text{with } y \text{ occurrences of 2.}$$

grows faster than Elementary Function Arithmetic can prove exists.

Elementary Function Arithmetic.

A concrete theory of arithmetic, in Hilbert's sense.

Hilbert: to say every number n is less than some prime p, is not concrete.

Concrete: every number n is less than some prime $p \leq n! + 1$.

$$\exists y < 2^{(x^2+2)} \, (\, x < y \; \wedge \; y \text{ is prime} \,)$$

Elementary function arithmetic uses $+\times, y^x$ and such bounds.